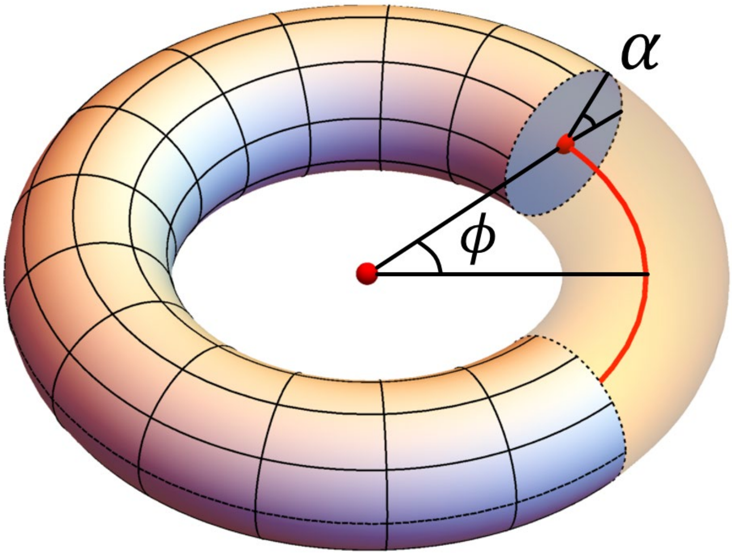
4CCP1300 Mathematics and Computation for Physics

Lecture notes
2018/19



Francisco J. Rodríguez Fortuño

King's College London

TABLE OF CONTENTS

1. VECTOR SPACES

1.1. VECTORS IN 3D: EQUATIONS OF LINES AND PLANES

A. DOT PRODUCT:

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

Projection of $\mathbf{v}$ in direction $\mathbf{u}$ is given by $v_u\hat{\mathbf{u}}$ with $v_u = \mathbf{v}\cdot\hat{\mathbf{u}}$ where $\hat{\mathbf{u}} = \mathbf{u}/\|\mathbf{u}\|$:

Length or norm of a vector: $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$

B. CROSS PRODUCT:

$$\mathbf{a}\times\mathbf{b} = \underbrace{\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta}_{\text{Area of parallelogram}}\ \hat{\mathbf{u}}$$

$\hat{\mathbf{u}}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, given by right hand rule

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\mathbf{x}}(a_y b_z - a_z b_y) - \hat{\mathbf{y}}(a_x b_z - a_z b_x) + \hat{\mathbf{z}}(a_y b_y - a_y b_x)$$

C. 3-D EQUATIONS FOR LINES AND PLANES

	Line	Plane
Parametric	$\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{v}$	$\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{u} + \mu\mathbf{v}$
Using given points a, b, c	$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})$	$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) + \mu(\mathbf{c}-\mathbf{a})$
	$\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b}$, with $\alpha+\beta = 1$	$\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$, with $\alpha+\beta+\gamma = 1$
Using products	$(\mathbf{r} - \mathbf{r}_0)\times\mathbf{v} = 0$	$(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{n} = 0$
Component equations (3D)	$\frac{x-x_0}{v_x} = \frac{y-y_0}{v_y} = \frac{z-z_0}{v_z}$	$n_x x + n_y y + n_z z = d$

D. EQUATION OF SPHERE

Sphere radius a, center $\mathbf{c}$: $\|\mathbf{r}_{\text{sphere}} - \mathbf{c}\| = a$

E. 3-D DISTANCES BETWEEN POINTS, LINES AND PLANES

Point $(\mathbf{p})$ to Line $(\mathbf{r}_0 + \lambda\hat{\mathbf{v}})$	$(\mathbf{p} - \mathbf{r}_{\text{line}}) \times \hat{\mathbf{v}}$
Point $(\mathbf{p})$ to Plane $(\mathbf{r} - \mathbf{r}_0) \cdot \hat{\mathbf{n}} = 0$	$\left(\mathbf{p} - \mathbf{r}_{\text{plane}}\right) \cdot \hat{\mathbf{n}}$
Line $(\mathbf{r}_1 + \lambda\mathbf{v}_1)$ to Line $(\mathbf{r}_2 + \lambda\mathbf{v}_2)$	$(\mathbf{r}_{\text{line1}} - \mathbf{r}_{\text{line2}}) \cdot \hat{\mathbf{n}}$ with $\hat{\mathbf{n}} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\lVert\mathbf{v}_1 \times \mathbf{v}_2\rVert}$
Line $(\mathbf{r}_1 + \lambda\mathbf{v})$ to Plane $(\mathbf{r} - \mathbf{r}_2) \cdot \hat{\mathbf{n}} = 0$	$\left(\mathbf{r}_{\text{line}} - \mathbf{r}_{\text{plane}}\right) \cdot \hat{\mathbf{n}}$ as long as $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$

F. PROPERTIES OF DOT AND CROSS PRODUCTS

Dot and cross products are linear operations:

$$(\alpha\mathbf{a} + \beta\mathbf{b}) \cdot \mathbf{v} = \alpha(\mathbf{a} \cdot \mathbf{v}) + \beta(\mathbf{b} \cdot \mathbf{v})$$

$$(\alpha\mathbf{a} + \beta\mathbf{b}) \times \mathbf{v} = \alpha(\mathbf{a} \times \mathbf{v}) + \beta(\mathbf{b} \times \mathbf{v})$$

Dot product is symmetric:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

Cross product is antisymmetric:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

PROBLEMS:

DOT AND CROSS PRODUCT

1) Show that if $\mathbf{a} = \mathbf{b} + \lambda\mathbf{c}$, then $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$

Sol:
From the distributive property under addition:

$$\mathbf{a} \times \mathbf{c} = (\mathbf{b} + \lambda\mathbf{c}) \times \mathbf{c} = (\mathbf{b} \times \mathbf{c}) + (\lambda\mathbf{c} \times \mathbf{c})$$

And since $(\mathbf{c} \times \mathbf{c}) = 0$, we arrive at the proof.

EQUATIONS OF LINES AND PLANES

2) Consider a line joining points $\mathbf{a}$ = (2,2,0) and $\mathbf{b}$ = (3,1,1). Write down the line in parametric form, using a cross product, and in component form.

Solution:

$\mathbf{r}_0 = (2,2,0)$

$\mathbf{v} = \mathbf{b} - \mathbf{a} = (3,1,1) - (2,2,0) = (1,-1,1)$

Parametric form: $\mathbf{r} = (2,2,0) + \lambda\,(1,-1,1)$

Using a cross product: $(\mathbf{r} - \mathbf{r}_0) \times \mathbf{v} = 0$

Component form:

$$\frac{x-2}{1} = \frac{y-2}{-1} = z$$

3) Consider a line joining points $\mathbf{a} = (1,2,0)$ and $\mathbf{b} = (3,1,0)$. Write down the line in parametric form, using a cross product, and in component form.

Solution:

$\mathbf{r}_0 = (1,2,0)$

$\mathbf{v} = \mathbf{b} - \mathbf{a} = (2,-1,0)$

Parametric form: $\mathbf{r} = (1,2,0) + \lambda(2,-1,0)$

Using a cross product: $(\mathbf{r} - \mathbf{r_0}) \times \mathbf{v} = 0 \rightarrow (\mathbf{r} - (1,2,0)) \times (2,-1,0) = 0$

Component form: $(x-1)/2 = (y-2)/-1 = (z-0)/0$. It cannot be written in component form, because the line is parallel to one of the coordinate planes!

DISTANCES

4) Find the distance between the lines

$$\frac{x-2}{2}=\frac{y-4}{1}=\frac{z-3}{2}$$

and

$$x-1=2-y=-z$$

Sol: A line crossing a point $\mathbf{r}_0=(x_0,y_0,z_0)$ with a direction $\mathbf{v}=(v_x,v_y,v_z)$ can be written in component equation as:

$$\frac{x-x_0}{v_x}=\frac{y-y_0}{v_y}=\frac{z-z_0}{v_z}$$

The first line includes a point $\mathbf{r}_1=\begin{pmatrix}2\\4\\3\end{pmatrix}$ and has direction $\mathbf{v}_1=\begin{pmatrix}2\\1\\2\end{pmatrix}$

The second line needs some re-ordering of terms to match the known form $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-0}{-1}$ and includes a point $\mathbf{r}_2=\begin{pmatrix}1\\2\\0\end{pmatrix}$ and has direction $\mathbf{v}_2=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$.

Given a point and a direction for each line, their distance is given by:

$$(\mathbf{r}_2-\mathbf{r}_1)\cdot\hat{\mathbf{n}} \qquad \text{with} \qquad \hat{\mathbf{n}}=\frac{\mathbf{v}_1\times\mathbf{v}_2}{\|\mathbf{v}_1\times\mathbf{v}_2\|}$$

$$\mathbf{n}=\mathbf{v}_1\times\mathbf{v}_2=(1,4,-3)^T$$

$$\hat{\mathbf{n}}=\mathbf{n}/\|\mathbf{n}\|=\left(\frac{1}{\sqrt{26}}\right)(1,4,-3)^T$$

is a unit vector joining the two lines at their closest approach.

A vector connecting the two lines is $(\mathbf{r}_2-\mathbf{r}_1)=(-1,-2,-3)^T$ and the distance between the two lines is given by the magnitude of the dot product $|(\mathbf{r}_2-\mathbf{r}_1)\cdot\hat{\mathbf{n}}|=\left|\begin{pmatrix}-1\\-2\\-3\end{pmatrix}\cdot\frac{1}{\sqrt{26}}\begin{pmatrix}1\\4\\-3\end{pmatrix}\right|=0$. The two lines intersect!

5) Calculate the minimum distance between the line $y = 2x + 3$ and the origin.

Solution:

It is not 3! Distance between point and line is: $d=\|(\mathbf{p}-\mathbf{r}_{\text{line}})\times\hat{\mathbf{v}}\|$.

To use this equation we need the line in parametric form $\mathbf{r}=\mathbf{r}_{\text{line}}+\lambda\mathbf{v}$

$\mathbf{r}_{\text{line}}$ is **any point in the line**, e.g., $(0,3)$

$\mathbf{v}$ is a vector in the direction of the line, we can find it as a vector joining any two points in the line, or directly by considering the meaning of the slope = 2, so that $\mathbf{v}=(1,2)$, which we then normalize into a unit vector:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(1,2)}{\sqrt{1^2+2^2}} = \frac{(1,2)}{\sqrt{5}}$$

Therefore, the distance between the origin and the line is, with $\mathbf{r}_{\text{line}}$ being **ANY point** in the line:

$$d = \|(\mathbf{p} - \mathbf{r}_{\text{line}}) \times \hat{\mathbf{v}}\|$$

$$= \left\|\big((0,0) - (0,3)\big) \times \frac{1}{\sqrt{5}}(1,2)\right\|$$

Cross products cannot be performed in 2D space!! But we can pretend that our working x-y plane is in 3D space at the plane z=0, for example, and all the distances remain the same

$$= \left\|\big((0,0,0) - (0,3,0)\big) \times \frac{1}{\sqrt{5}}(1,2,0)\right\|$$

$$= \frac{1}{\sqrt{5}}\|(0,-3,0) \times (1,2,0)\| = \frac{3}{\sqrt{5}}$$

6) Give the coordinates of the point at which the line $\mathbf{r} = (1,3) + \lambda(1,1)$ is closest to the point $(1,1)$.

Sol: This is not only asking for the minimum distance, which is easily calculated with a known equation:

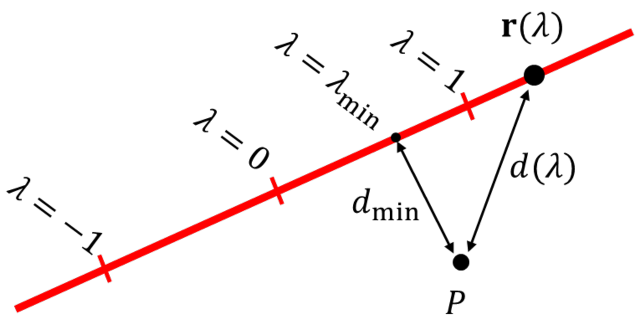
$$d_{\min} = \|(\mathbf{p} - \mathbf{r}_{\text{line}}) \times \hat{\mathbf{v}}\| = \left\|\big((1,1) - (1,3)\big) \times \frac{(1,1)}{\sqrt{1+1}}\right\| = \frac{1}{\sqrt{2}}\|(0,2) \times (1,1)\| = \sqrt{2}$$

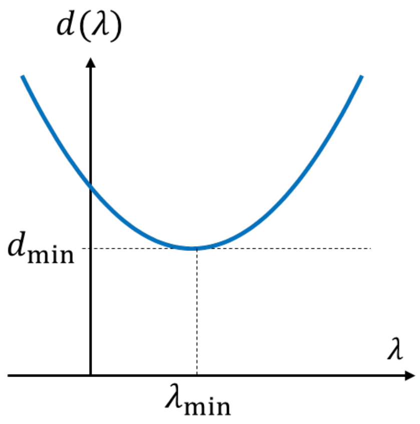
But we are also being asked **the point at which the distance is minimal!**

To solve this, we need to think a bit. We can find the distance from ANY point in the line as a function of the parameter λ and then equate it to the known minimum distance to solve for λ:

$$d(\lambda) = \|\mathbf{p} - \mathbf{r}_{\text{line}}(\lambda)\| = d_{\min}$$

Solve quadratic equation for $\lambda_{\min}$.
Quadratic equation must have **only one (double) root**: ($\lambda_{\min} = -b/2a$)





Therefore, we do:

$$\begin{aligned}
d(\lambda) = \|\mathbf{r}_{\text{line}}(\lambda) - \mathbf{p}\| &= d_{\min} \\
\|(1,3) + \lambda(1,1) - (1,1)\| &= d_{\min} \\
\|(0,2) + \lambda(1,1)\| &= d_{\min} \\
\left\|\begin{pmatrix}\lambda \\ 2+\lambda\end{pmatrix}\right\| &= d_{\min} \\
\sqrt{\lambda^2 + (2+\lambda)^2} &= d_{\min} \\
\sqrt{\lambda^2 + 4 + 4\lambda + \lambda^2} &= d_{\min} \\
\sqrt{2\lambda^2 + 4\lambda + 4} &= d_{\min}
\end{aligned}$$

Squaring both sides:

$$2\lambda^2 + 4\lambda + 4 = d_{\min}^2$$

Substituting $d_{\min} = \|(\mathbf{r}_{\text{line}} - \mathbf{p}) \times \hat{\mathbf{v}}\| = \sqrt{2}$

$$2\lambda^2 + 4\lambda + 2 = 0$$

We can solve this quadratic equation, and it should have only one solution! Since we know it should have only one solution, we know that the discriminant (the $\sqrt{b^2 - 4ac}$ part) must be zero, so the quadratic formula equation is really easy $\lambda_{\min} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-b}{2a} = \frac{-4}{4} = -1$. In fact, we don't need to find $d_{\min}^2$ at all, because the solution $\frac{-b}{2a}$ is not affected by adding or subtracting $d_{\min}^2$.

Finally, the point $\lambda = -1$ corresponds to $\mathbf{r}_{\text{line}}(\lambda = -1) = (1,3) - 1(1,1) = (0,2)$.

Notice that vector formulas (the calculation of $d_{\min}$) has "saved" us from having to find derivatives to find the local minimum (which would be an alternative way of doing it).

7) Calculate the point at which the line $y = 2x + 3$ is closest to the origin.

Tip: In a previous problem we found $y = 2x + 3 \;\rightarrow\; \mathbf{r}(\lambda) = (0,3) + \lambda(1,2)$

Solution: Therefore, we can follow the procedure for finding the value of λ which achieves that distance as above:

$$\begin{aligned}
d(\lambda) = \|\mathbf{r}_{\text{line}}(\lambda) - \mathbf{p}\| &= d_{\min} \\
\|(0,3) + \lambda(1,2) - (0,0)\| &= d_{\min} \\
\left\|\begin{pmatrix}\lambda \\ 3+2\lambda\end{pmatrix}\right\| &= d_{\min} \\
\sqrt{\lambda^2 + (3+2\lambda)^2} &= d_{\min} \\
\sqrt{\lambda^2 + 9 + 12\lambda + 4\lambda^2} &= d_{\min} \\
\sqrt{5\lambda^2 + 12\lambda + 9} &= d_{\min} \\
5\lambda^2 + 12\lambda + 9 &= d_{\min}^2
\end{aligned}$$

We can solve this quadratic equation for $\lambda_{\min}$. Since we know that $d_{\min}$ occurs exactly at the minimum, the solution is given by:

$$\lambda_{\min} = \frac{-b}{2a} = \frac{-12}{10} = -\frac{6}{5}$$

So that the point of minimum distance is:

$$\mathbf{r}\left(\lambda = -\frac{6}{5}\right) = (0,3) - \frac{6}{5}(1,2) = \left(-\frac{6}{5}, \frac{3}{5}\right)$$

Double check your answer, check that the distance from that point to the origin is indeed $\frac{3}{\sqrt{5}}$.

8) Consider the line segment $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{v}$ with λ in the interval [0,2] (i.e. it is not an infinite line, only a segment of it). Come up with an algorithm to find the minimum distance between this line segment and a point $\mathbf{p}$, without requiring differentiation.

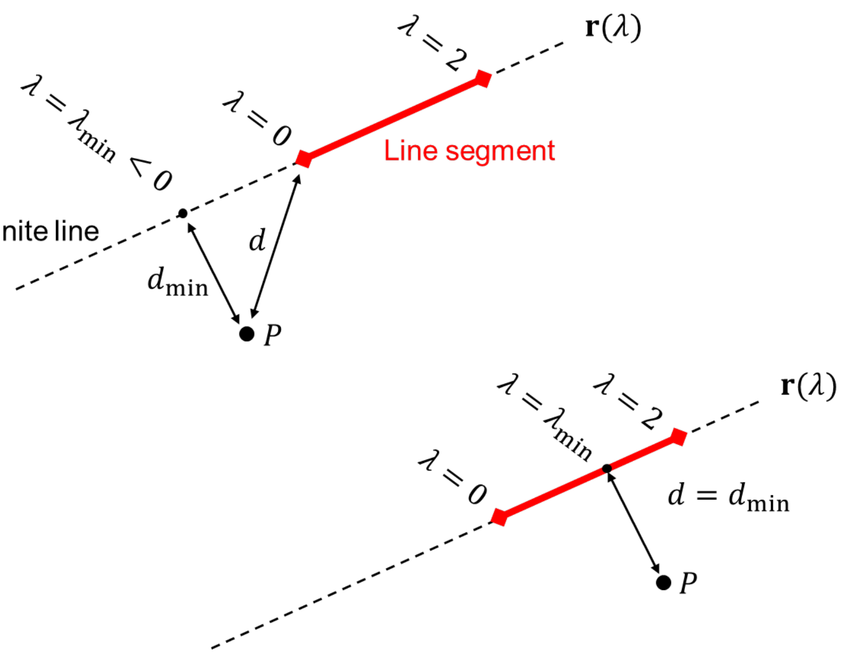
First find the distance between the infinite line $\mathbf{r} = \mathbf{r_0} + \lambda\mathbf{v}$ and the point $\mathbf{p}$, lets call it $d_{\min} = \left\|(\mathbf{p} - \mathbf{r_0}) \times \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\right\|$

Find the value of lambda for which this minimum distance takes place, by solving the equation:

$\|\mathbf{r}_{\text{line}}(\lambda) - \mathbf{p}\| = d_{\min}$

$\|\mathbf{r}_0 + \lambda\mathbf{v} - \mathbf{p}\| = d_{\min}$

This will give rise to a quadratic equation in λ, with only ONE (double) solution $\lambda_{\min}$. If the value of $\lambda_{\min}$ is outside the range of the line segment [0,2], then the end of the line segment closest in λ to the solution $\lambda_{\min}$ will be the point with smallest distance from the line segment to $\mathbf{p}$. If the value is inside the range, then the minimum distance will be $d_{\min}$.



9) Find the distance from the point P with coordinates (1,2,3) to the plane that contains the points A, B, C having coordinates (0,1,0), (2,3,1) and (5,7,2)

Sol:

Distance from a point P to a plane π is given by: $(\mathbf{p} - \mathbf{r}_{\text{plane}}) \cdot \hat{\mathbf{n}}$. Where $\mathbf{r}_{\text{plane}}$ is **any point** in the plane: e.g. A, B or C, let's take $\mathbf{r}_{\text{plane}} = \mathbf{a}$

And we also need the normal to the plane $\hat{\mathbf{n}}$.

The normal to the plane can be calculated via the cross product of any two vectors in the plane

$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$
$$= (2-0, 3-1, 1-0) \times (5-0, 7-1, 2-0)$$
$$= (2,2,1) \times (5,6,2)$$
$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 2 & 1 \\ 5 & 6 & 2 \end{vmatrix} = \hat{\mathbf{x}}(4-6) - \hat{\mathbf{y}}(4-5) + \hat{\mathbf{z}}(12-10) = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Remember we need the normalized unit vector $\hat{\mathbf{n}}$.

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{(-2,1,2)}{\sqrt{(-2)^2 + (1)^2 + (2)^2}} = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

So, we can finally calculate the distance:

$$d = |(\mathbf{p} - \mathbf{r}_{\text{plane}}) \cdot \hat{\mathbf{n}}|$$
$$= \left| \left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \cdot \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right|$$
$$= \frac{1}{3}\left| \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right| = \frac{1}{3}(-2+1+6) = \frac{5}{3}$$

INTERSECTIONS

10) Find the direction of the line of intersection of the two planes $x + 3y - z = 5$ and $2x - 2y + 4z = 3$.

Sol:
$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$

$= (1,3,-1) \times (2,-2,4)$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 3 & -1 \\ 2 & -2 & 4 \end{vmatrix} = \hat{\mathbf{x}}\big(3 \cdot 4 - (-1)\cdot(-2)\big) - \hat{\mathbf{y}}(1 \cdot 4 - (-1) \cdot 2) + \hat{\mathbf{z}}(1 \cdot (-2) - 3 \cdot 2)$$

$$= \begin{pmatrix} 10 \\ -6 \\ -8 \end{pmatrix}$$

11) Find the equation in parametric form for the line of intersection of the two planes

$x + 3y - z = 5$ and $2x - 2y + 4z = 3$.

Solution: The planes are the same as in the previous problem, the vector parallel to the line is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (1,3,-1) \times (2,-2,4) = (10,-6,-8)$. For the parametric equation $\mathbf{r}(\lambda) = \mathbf{r}_0 + \lambda\mathbf{v}$ we need to find any point of the line $\mathbf{r}_0$.

To find a point of the line, we must find **a point that exists simultaneously** in both planes $x + 3y - z = 5$ and $2x - 2y + 4z = 3$, in other words, we need to solve the two equations simultaneously and find any solution. There are lots of ways to do this. Let’s subtract two times the first equation to the second equation:

$$(2x - 2y + 4z) - 2(x + 3y - z) = 3 - 2(5)$$
$$-8y + 6z = -7$$

This is, in fact, one of the two equations which would define the line of intersection. We don’t need to find the other, as we only need any point. For example, we can set $y_0 = 0$ and find $z_0 = -\frac{7}{6}$. Now we can find a value for x_0 from the two equations of the planes. We can use $2x - 2y + 4z = 3$, substituting y_0 and z_0 we find $2x_0 - \frac{7}{6}4 = 3 \;\rightarrow\; x_0 = \frac{3}{2} + \frac{28}{12} = \frac{9}{6} + \frac{14}{6} = \frac{23}{6}$. We can check that the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ also fulfils the equation for the second plane. Therefore, we have found a point on the line, and we have all the required information:

$$\mathbf{r}(\lambda) = \mathbf{r}_0 + \lambda\mathbf{v}$$
$$= \begin{pmatrix} 23/6 \\ 0 \\ -7/6 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ -6 \\ -8 \end{pmatrix}$$

12) Find the parametric equation of the line of intersection of the two planes given by $x - 5 = 0$ and $y = z$.

Sol:
Do not be confused by the notation,

$x - 5 = 0$ is the plane $x + 0y + 0z = 5$
$y = z$ is the plane $0x + y - z = 0$.

$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$

$= (1,0,0) \times (0,1,-1)$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\mathbf{x}}(0) - \hat{\mathbf{y}}(1 \cdot 1 - 0) + \hat{\mathbf{z}}\big((-1) - 0\big) = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

Now find any point r_0 simultaneous solution to the two planes: ANY point will do. For example, $x = 5$, $y = 0$, $z = 0$. So, the equation of the line is (notice I used $-\mathbf{v}$ because it looked simpler and is equally valid as a vector parallel to the line)

$$\mathbf{r}(\lambda) = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(INCLUDING SPHERES MAKES EQUATIONS BECOME QUADRATIC)

13) Consider the line $\mathbf{r} = (2,2) + \lambda(2,1)$. This line intersects a circle with radius $r = \sqrt{2}$ and center $\mathbf{c} = (-1,0)$ at two points $\mathbf{a}$ and $\mathbf{b}$. Find their coordinates.

Sol:

Equation for the line: $\mathbf{r} = \mathbf{r_0} + \lambda\mathbf{v}$
Equation for the circle: $\|\mathbf{r} - \mathbf{c}\| = r$

Substituting one into the other we can solve for the values of λ that fulfil both conditions simultaneously:
$\|\mathbf{r_0} + \lambda\mathbf{v} - \mathbf{c}\| = r$
$\|(2,2) + \lambda(2,1) - (-1,0)\| = \sqrt{2}$
$\|(3 + 2\lambda, 2 + \lambda)\| = \sqrt{2}$
Evaluating the norm and squaring both sides:
$(3 + 2\lambda)^2 + (2 + \lambda)^2 = 2$
$(9 + 12\lambda + 4\lambda^2) + (4 + 4\lambda + \lambda^2) = 2$
$5\lambda^2 + 16\lambda + 11 = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-16 \pm \sqrt{256 - 220}}{10} = \frac{-16 \pm 6}{10} = \{-1, -11/5\}$$

Therefore, the two points of intersection are found by substituting λ into $\mathbf{r} = \mathbf{r_0} + \lambda\mathbf{v}$:

$\mathbf{a} = \mathbf{r_0} + (-1)\mathbf{v} = (2,2) - (2,1) = (0,1)$

$\mathbf{b} = \mathbf{r_0} + (-11/5)\mathbf{v} = (2,2) - (11/5)(2,1) = \left(-\frac{12}{5}, -\frac{1}{5}\right)$

14) Find the minimum distance between a point with position vector $\mathbf{p}$, and the surface of a sphere with radius a and centre at position vector $\mathbf{c}$.

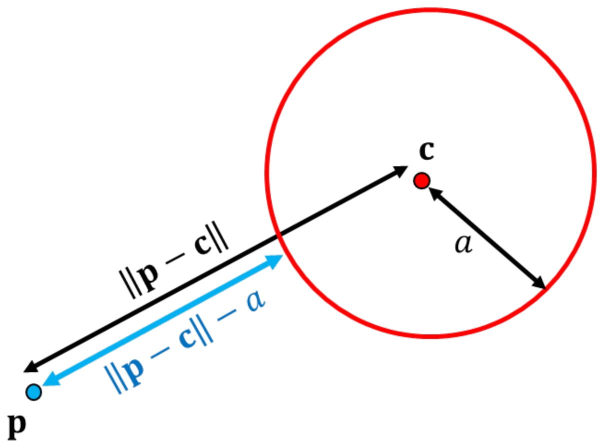
Sol:

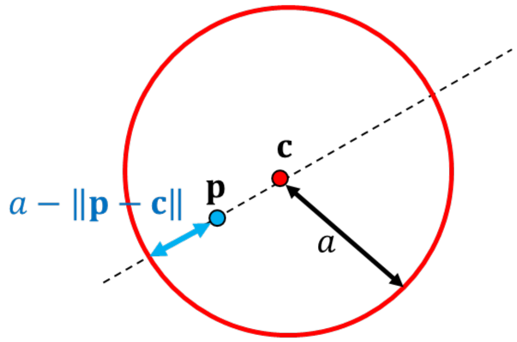
First find the distance $\|\mathbf{p} - \mathbf{c}\|$

If $\|\mathbf{p} - \mathbf{c}\| > a$, then the required distance is decreased by an amount a, so that $d = \|\mathbf{p} - \mathbf{c}\| - a$.

But what if $\|\mathbf{p} - \mathbf{c}\| < a$? Then we overshoot the "subtraction" as we get a negative number, but this negative number is still the distance! We just need to take the absolute value:

So, finally: $d = |\|\mathbf{p} - \mathbf{c}\| - a|$

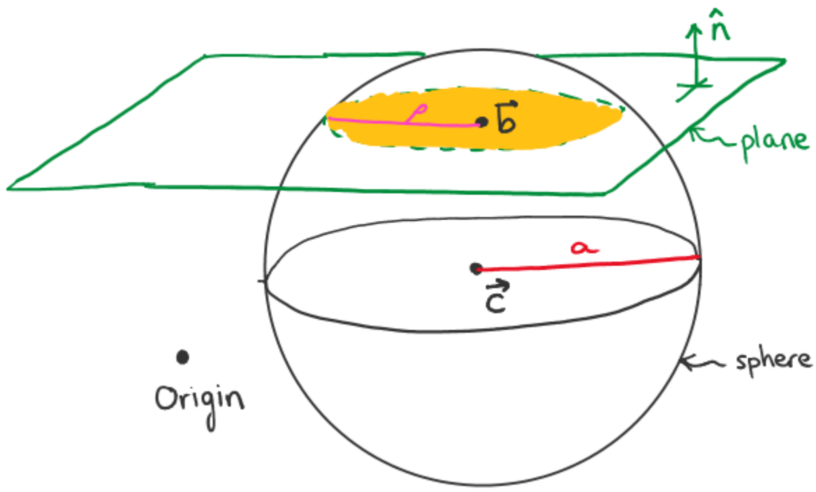




Distance from $\|\mathbf{r}_{\text{sphere}} - \mathbf{c}\| = a$ to point $\mathbf{p}$: $\quad d_{\min} = |\|\mathbf{p} - \mathbf{c}\| - a|$

15) Find the radius ρ of the circle that is the intersection of the plane $\hat{\mathbf{n}} \cdot \mathbf{r} = p$ and the sphere of radius a centred on the point with position vector $\mathbf{c}$. The answer must be given in terms of the known parameters only ($\hat{\mathbf{n}}$, p, a and $\mathbf{c}$)

Sol:



Let's define $\mathbf{b}$ as the centre of the desired circle of intersection
The equation of the desired circle is $|\mathbf{r} - \mathbf{b}| = \rho$, with $\mathbf{r}$ constrained to the plane $\hat{\mathbf{n}} \cdot \mathbf{r} = p$.

We can visualize that:

$\mathbf{b} - \mathbf{c}$ must be parallel to $\hat{\mathbf{n}}$
therefore $\mathbf{b} - \mathbf{c} = \lambda \hat{\mathbf{n}}$

If you have trouble visualizing why this is, think of the cone formed with tip at $\mathbf{c}$ and base equal to the intersection circle. The base of that cone is within the plane, with normal $\hat{\mathbf{n}}$. Clearly the center of the plane surface of the cone will be the center of the circle, point $\mathbf{b}$, and from this point, the tip of the cone can be reached by moving perpendicularly to the base.

From Pythagoras' theorem we have:
$||\mathbf{b} - \mathbf{c}||^2 + \rho^2 = a^2$

So $\rho^2 = a^2 - ||\mathbf{b} - \mathbf{c}||^2$

$\rho = \sqrt{a^2 - ||\mathbf{b} - \mathbf{c}||^2}$

Now we just need to find the length of the vector $\mathbf{b} - \mathbf{c}$. Since vector $\mathbf{b} - \mathbf{c}$ is perpendicular to the plane, its length corresponds to the distance between point $\mathbf{c}$ and the plane $\hat{\mathbf{n}} \cdot \mathbf{r} = p$.

$||\mathbf{b} - \mathbf{c}|| =$ distance between point $\mathbf{c}$ and the plane $\hat{\mathbf{n}} \cdot \mathbf{r} = p$

The distance between a point and a plane is $(\mathbf{c} - \mathbf{r}_{\text{plane}}) \cdot \hat{\mathbf{n}}$
We need any point in the plane:

The equation of the plane is $\mathbf{r} \cdot \hat{\mathbf{n}} = p$

So any possible point in the plane $\mathbf{r}_{\text{plane}}$ is any point which fulfills $\mathbf{r}_{\text{plane}} \cdot \hat{\mathbf{n}} = p$, for example we can take $\mathbf{r}_{\text{plane}} = p\hat{\mathbf{n}}$

Therefore, the required distance is $||\mathbf{b} - \mathbf{c}|| = |(\mathbf{c} - p\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}}| = |\mathbf{c} \cdot \hat{\mathbf{n}} - p|$

So finally we have:

$$\rho = \sqrt{a^2 - |\mathbf{c} \cdot \hat{\mathbf{n}} - p|^2}$$

There are several conditions for this to be a real number (meaning that the sphere does intersect with the plane)

1.2 N-DIMENSIONAL VECTOR SPACES

A. BASIS, SPAN, LINEAR DEPENDENCE AND INDEPENDENCE

Span:

$\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\} \equiv$ Space of all vectors reached by linear combinations $\mathbf{v} = \sum_i a_i \mathbf{v}_i$

Linear dependence/independence:

$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ are linearly independent $\Leftrightarrow$ $\sum_i a_i \mathbf{v}_i \neq 0$ always except for trivial case $a_i = 0$.
$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ are linearly dependent $\Leftrightarrow$ $\sum_i a_i \mathbf{v}_i = 0$ for some non-zero values of a_i
$\Leftrightarrow$ at least one of the $\mathbf{v}_i$ can be written as a linear combination of the others.

Dimensions of span:

$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ are linearly independent $\Leftrightarrow \dim[\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}] = N$
$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ are linearly dependent $\Leftrightarrow \dim[\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}] \leq N$

Basis:

$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_\mathrm{N}\}$ is a basis of space $\mathcal{V} \Leftrightarrow \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_\mathrm{N}\}$ are linearly independent vectors which span the whole of $\mathcal{V}$ whose dimension is therefore $\dim[\mathcal{V}] = N$

B. N-DIMENSIONAL VECTOR SPACES

A vector space $\mathcal{V}$ is not necessarily used to represent geometrical vectors. It can represent any set whose elements behave like vectors, that is, fulfils the following axioms:

- Vector addition:
  - Closure: $\mathbf{u} + \mathbf{v}$ is in $\mathcal{V}$
  - Commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - Associative: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - Additive identity: Vector $\mathbf{0} \in \mathcal{V}$ exists such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  - Additive inverse: Vector $(-\mathbf{u}) \in \mathcal{V}$ exists such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for every $\mathbf{u}$

- Multiplication by a scalar:
  - Closure: $\lambda\mathbf{u}$ is in $\mathcal{V}$
  - Distributive over vector addition: $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$
  - Distributive over scalar addition: $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$
  - Associative: $(\lambda\mu)\mathbf{v} = \lambda(\mu\mathbf{v})$
  - Scalar identity: Scalar 1 exists such that $1\mathbf{v} = \mathbf{v}$

Examples: RGB colour space, polynomials of degree n, functions $f(x)$ on an interval $[a, b]$…

C. INNER PRODUCT

Axioms of (Hermitian) inner product:

- (Conjugate) Symmetry $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle^*$
- Positive definiteness $\langle \mathbf{a}, \mathbf{a} \rangle \geq 0$ (only equal to 0 if $||\mathbf{a}|| = 0$)
- Linearity in 1st argument* which implies (conjugate) linearity in 2nd

$$\langle \alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{c} \rangle = \alpha\langle \mathbf{a}, \mathbf{c} \rangle + \beta\langle \mathbf{b}, \mathbf{c} \rangle$$
$$\langle \mathbf{c}, \alpha\mathbf{a} + \beta\mathbf{b} \rangle = \alpha^*\langle \mathbf{c}, \mathbf{a} \rangle + \beta^*\langle \mathbf{c}, \mathbf{b} \rangle$$

Definitions associated to inner product:

- Norm of a vector: $||\mathbf{a}|| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$
- Orthogonality between vectors: $\langle \mathbf{a}, \mathbf{b} \rangle = 0 \quad \Leftrightarrow \quad \mathbf{a}$ and $\mathbf{b}$ are orthogonal

* Remember that linearity in 1st or 2nd argument is an arbitrary choice. A typical choice in physics is to have linearity in the 2nd argument and conjugate linearity in the first. This alternative definition of inner product is typically written as $\langle \mathbf{a} | \mathbf{b} \rangle$

Useful applications of the inner product:

Orthogonal and orthonormal basis:

$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_\mathrm{N}\}$ orthogonal basis $\Leftrightarrow \langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ if $i \neq j$

$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \cdots, \hat{\mathbf{e}}_\mathrm{N}\}$ orthonormal basis $\Leftrightarrow \langle \hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j \rangle = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} = \delta_{ij}$ Kronecker delta

Components of a vector in orthogonal (or orthonormal) basis:

Components of a vector: $v_i = \frac{\langle \mathbf{v}, \mathbf{e}_i \rangle}{\langle \mathbf{e}_i, \mathbf{e}_i \rangle}$ if $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_\mathrm{N}\}$ is orthogonal basis

Components of a vector: $v_i = \langle \mathbf{v}, \hat{\mathbf{e}}_i \rangle$ if $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \cdots, \hat{\mathbf{e}}_\mathrm{N}\}$ is orthonormal basis

Projection of a vector into a subspace:

Space V; Subspace $W = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_\mathrm{N}\}$ orthogonal basis; $\mathbf{w} \in W$ closest vector to $\mathbf{v} \in V$

$$\mathbf{w} = \frac{\langle \mathbf{v}, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle}\mathbf{e}_1 + \frac{\langle \mathbf{v}, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle}\mathbf{e}_2 + \cdots + \frac{\langle \mathbf{v}, \mathbf{e}_N \rangle}{\langle \mathbf{e}_N, \mathbf{e}_N \rangle}\mathbf{e}_N$$

(but only if $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_\mathrm{N}\}$ is orthogonal basis)

Hermitian inner product of vectors in terms of orthonormal basis components:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum a_i b_i^*$$

$\mathbf{a} = \sum_i a_i \hat{\mathbf{e}}_i$ and $\mathbf{b} = \sum_i b_i \hat{\mathbf{e}}_i$ with $\{\hat{\mathbf{e}}_i\}$ **orthonormal** basis.

D. FUNCTION VECTOR SPACES

The set of functions $f(x)$ in an interval $[a, b]$ defines an (infinite-dimensional) **vector space**.

Hermitian inner product for function vector space:

$$\langle f(x), g(x)\rangle = \int_a^b f(x) g^*(x) \mathrm{d}x$$

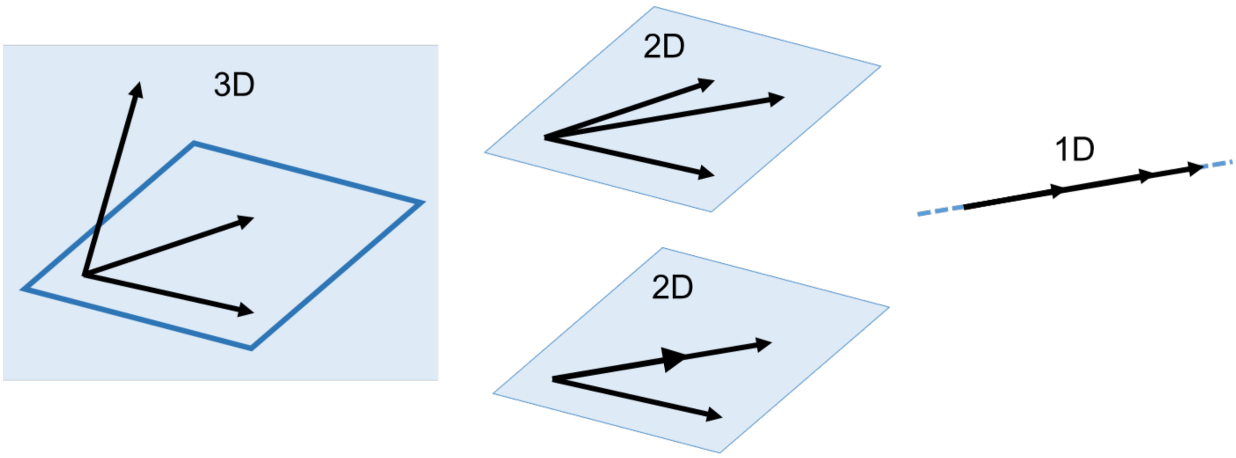
With this definition we can now calculate **norms**, **distances** and **projections** between functions.

Possible basis for this vector space: "Delta functions" of position, coefficients of x (i.e. a Taylor expansion), etc…

PROBLEMS:

UNDERSTANDING SPAN, LINEAR DEPENDENCE AND INDEPENDENCE

We know that $\mathrm{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ can have 0, 1, 2 or 3 dimensions. It will have 3 dimensions only if the vectors are linearly independent. It will have 0, 1 or 2 dimensions if the vectors are linearly dependent.



How can we mathematically tell the different cases apart? Let's see examples of them:

1) Check the linear independence of vectors $\mathbf{v}_1 = (1,1,0)$, $\mathbf{v}_2 = (1,0,1)$ and $\mathbf{v}_3 = (0,1,1)$

Sol:

Linear independence: The equation $a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} = \mathbf{0}$ has no solutions apart from the trivial one. Write down the actual vectors and write the equations component by component:

$$a_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + a_2 \begin{pmatrix}1\\0\\1\end{pmatrix} + a_3 \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases} a_1 + a_2 = 0 & (Eq.1) \\ a_1 + a_3 = 0 & (Eq.2) \\ a_2 + a_3 = 0 & (Eq.3) \end{cases}$$

Solve for the coefficients:

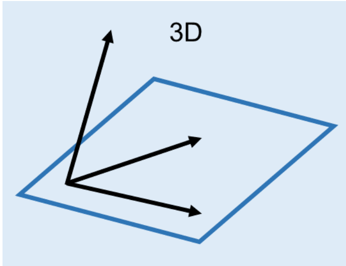
(1)-(2) $\rightarrow a_2 - a_3 = 0 \rightarrow a_3 = a_2$

Into (3) $a_2 + a_3 = 0 \rightarrow a_2 + a_2 = 0 \rightarrow a_2 = a_3 = 0$

Into (1) $a_1 = -a_2 = 0$

The only solution is the trivial one $a_1 = a_2 = a_3 = 0$.

Therefore, the vectors are linearly independent



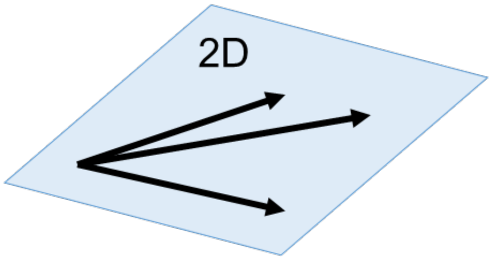
2) Check the linear dependence of vectors $\mathbf{v}_1 = (1,1,0)$, $\mathbf{v}_2 = (1,0,1)$ and $\mathbf{v}_3 = (0,1,-1)$. How can we mathematically check the number of vectors which are redundant in terms of the span?

Linear dependence: The equation $a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} = \mathbf{0}$ has solutions different from the trivial one.

Write down the actual vectors and write the equations component by component:

$$a_1\begin{pmatrix}1\\1\\0\end{pmatrix} + a_2\begin{pmatrix}1\\0\\1\end{pmatrix} + a_3\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases} a_1 + a_2 = 0 & (Eq.1) \\ a_1 + a_3 = 0 & (Eq.2) \\ a_2 - a_3 = 0 & (Eq.3) \end{cases}$$



Solve for the coefficients:

(1)-(2) $\rightarrow a_2 - a_3 = 0 \rightarrow a_3 = a_2$

Into (3) $a_3 - a_3 = 0 \rightarrow$ True for any arbitrary value $a_3 = \lambda \rightarrow a_2 = \lambda$

Into (1) $\rightarrow a_1 + a_3 = 0 \rightarrow a_1 + \lambda = 0 \rightarrow a_1 = -\lambda$

Therefore, the general solution to the system is:

$$\{a_1 = -\lambda, a_2 = \lambda, a_3 = \lambda\}$$

Therefore, the vectors are linearly dependent.

The solution has 1 degree of freedom. Interestingly notice that we can write the general solution as the equation of a line, in the parameter space of $(a_1, a_2, a_3) = \lambda \cdot (-1,1,1)$.

When the solution to $\sum a_i\mathbf{v}_i = 0$ has N degrees of freedom, it means that N of the vectors are redundant in terms of the span (and can be removed)

In this case, one of the three vectors is redundant in terms of the span.

The $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is therefore 2-dimensional (a plane)

This will be clarified when we study the “theorem of dimensions” in the matrices lecture.

i.e. if we remove one of the vectors, being a linear combination of the others, the remaining two will be independent.

3) Check the linear dependence of vectors $\mathbf{v}_1 = (0, -1, 1)$, $\mathbf{v}_2 = (0, 1, -1)$ and $\mathbf{v}_3 = (0, 2, -2)$. State the dimension of their span, and state which vectors could be removed from the set without affecting the span.

By inspection we can immediately see that the three vectors are parallel, in the same line. So, a quick answer could be: the span is one dimensional, we can remove any two of the three vectors.

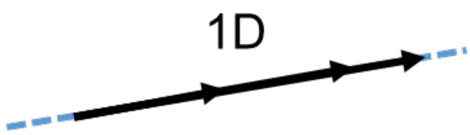
But it is interesting to see how this affects the solutions of the equation $\sum a_i \mathbf{v}_i = 0$

$$a_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = 0 & (Eq. 1) \\ -a_1 + a_2 + 2a_3 = 0 & (Eq. 2) \\ a_1 - a_2 - 2a_3 = 0 & (Eq. 3) \end{cases}$$

From (2) → $a_1 = a_2 + 2a_3$
Into (3) → $0 = 0$

What's going on? What happens is that any combination in which $a_1 = a_2 + 2a_3$ is a valid solution. Therefore, we have two degrees of freedom, we can choose $a_2 = \lambda$, $a_3 = \mu$, and $a_1 = \lambda + 2\mu$. The general solution to the system can be written as the **parametric equation of a plane** in the **space of coefficients** $(a_1, a_2, a_3) = \lambda(1,1,0) + \mu(2,0,1)$.



The solution has 2 degrees of freedom, which means that two of the three vectors are redundant in terms of the span. The $\mathrm{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is (3 vectors – 2 degrees of freedom of the linear independence equation) = 1-dimensional line.

In this case, any two of the vectors could be removed without affecting the span.

4) Check the linear dependence of vectors $\mathbf{v}_1 = (0,-1,1)$, $\mathbf{v}_2 = (0,1,-1)$ and $\mathbf{v}_3 = (0,1,2)$. State the dimension of their span, and state which vectors could be removed from the set without affecting the span.

This time, the situation is a bit trickier. It is clear that $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, so they count as one when defining a span. Vector $\mathbf{v}_3$ is independent of either $\mathbf{v}_1$ or $\mathbf{v}_2$. This all means that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent, but we could say that the blame for that falls more heavily on $\mathbf{v}_1$ and $\mathbf{v}_2$ than on $\mathbf{v_3}$. In fact, $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ will be a two-dimensional plane, and we can remove either $\mathbf{v}_1$ or $\mathbf{v}_2$ without affecting the span. However, if we choose wrongly and decided to remove $\mathbf{v_3}$, then the span would lose one additional dimension! How is all this reflected on the maths?

$a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} = \mathbf{0}$

$$a_1\begin{pmatrix}0\\-1\\1\end{pmatrix} + a_2\begin{pmatrix}0\\1\\-1\end{pmatrix} + a_3\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

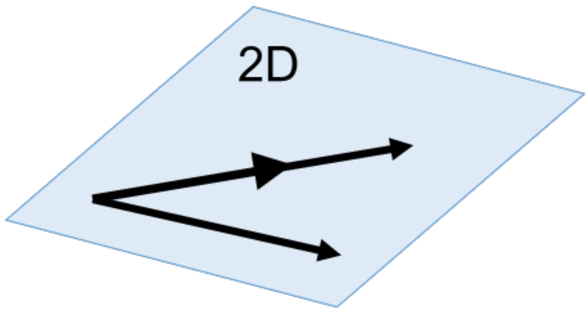
$$\begin{cases} 0 = 0 & (Eq.1) \\ -a_1 + a_2 + a_3 = 0 & (Eq.2) \\ a_1 - a_2 + 2a_3 = 0 & (Eq.3) \end{cases}$$

Adding (2)+(3) we get $a_3 = 0$.

Substituting this into the other two equations just tells us that $a_1 = a_2 = \lambda$, a free parameter.

The general solution can be written in vector form as $(a_1, a_2, a_3) = \lambda(1,1,0)$ and has one degree of freedom, so that the $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a (3 vectors – 1 degrees of freedom of the linear independence equation) = 2-dimensional plane.

Maths is also telling us that vector 3 is special because the linear combinations that determine the linear dependence always have $a_3 = 0$ and thus never involve $\mathbf{v_3}$. Indeed, $\mathbf{v_3}$ is more important than the other two, in terms of their span.



These cases might seem trivial in this 3D case, but hopefully you can realize how understanding these results could be useful for understanding the span of a set of vectors in higher-dimensional spaces.

5) Let's deal with 4-dimensional spaces. Check the linear dependence of vectors $\mathbf{v}_1 = (-1,1,1,1)$, $\mathbf{v}_2 = (1,-1,1,1)$, $\mathbf{v}_3 = (1,1,-1,1)$ and $\mathbf{v}_4 = (1,1,1,1)$.

Sol:

Linear independence: The equation $a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} + a_4\mathbf{v_4} = \mathbf{0}$ has no solutions apart from the trivial one.

Write down the actual vectors and write the equations component by component:

$$a_1\begin{pmatrix}-1\\1\\1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\-1\\1\\1\end{pmatrix} + a_3\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} + a_4\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{cases}-a_1 + a_2 + a_3 + a_4 = 0 & (Eq.1)\\ a_1 - a_2 + a_3 + a_4 = 0 & (Eq.2)\\ a_1 + a_2 - a_3 + a_4 = 0 & (Eq.3)\\ a_1 + a_2 + a_3 + a_4 = 0 & (Eq.4)\end{cases}$$

Solve for the coefficients:

$$(1)+(2) \rightarrow a_3 + a_4 = 0 \rightarrow a_3 = -a_4$$

$$(2)+(3) \rightarrow a_1 + a_4 = 0 \rightarrow a_1 = -a_4$$

$$(3)+(4) \rightarrow a_1 + a_2 + a_4 = 0 \rightarrow a_2 = -a_1 - a_4 = 0$$

Substituting the previous 3 results into (1): $-a_1 + a_2 + a_3 + a_4 = 0 \rightarrow a_4 - a_4 + a_4 = 0 \rightarrow a_4 = 0$

And therefore $a_1 = a_3 = -a_4 = 0$

The only solution is the trivial one $a_1 = a_2 = a_3 = a_4 = 0$.

Therefore, the vectors are linearly independent

EXPANDING A VECTOR IN A BASIS

6) Expand the vector $\mathbf{v} = (-1,1,1)$ in the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ with $\mathbf{e_1} = (1,0,1)$, $\mathbf{e_2} = (0,1,1)$, and $\mathbf{e_3} = (1,1,0)$.

First check that the vectors in the basis are linearly independent, so that we can expand them.

Now we need to solve for the linear coefficients a_1, a_2, a_3 such that:

$$\mathbf{v} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$$

Substitute the actual vectors and write the equations component by component:

$$\begin{pmatrix}-1\\1\\1\end{pmatrix} = a_1\begin{pmatrix}1\\0\\1\end{pmatrix} + a_2\begin{pmatrix}0\\1\\1\end{pmatrix} + a_3\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\begin{cases}-1 = a_1 + a_3 & (Eq.1)\\ 1 = a_2 + a_3 & (Eq.2)\\ 1 = a_1 + a_2 & (Eq.3)\end{cases}$$

Solve for the coefficients:

From (1), $a_3 = -a_1 - 1$

From (2), $a_2 = 1 - a_3$, and substituting the result from (1) $a_2 = 2 + a_1$

Substituting into (3), $1 = a_1 + 2 + a_1 \;\rightarrow\; 2a_1 = -1 \rightarrow\; a_1 = -\frac{1}{2}$

Therefore $a_2 = 2 + a_1 = \frac{3}{2}$

Into (1), $a_3 = -a_1 - 1 = -\frac{1}{2}$

So finally $\mathbf{v} = -\frac{1}{2}\mathbf{e_1} + \frac{3}{2}\mathbf{e_2} - \frac{1}{2}\mathbf{e_3}$

We could even write $\mathbf{v} = \frac{1}{2}\begin{pmatrix}-1\\3\\-1\end{pmatrix}$ in basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$

7) Find the coordinates a_1, a_2, a_3 of the vector $\mathbf{v} = (1,2,3)$ with respect to the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ given by vectors $\mathbf{e_1} = (1,1,0)$, $\mathbf{e_2} = (1,0,1)$ and $\mathbf{e_3} = (1,1,1)$.

Sol:

First check that the vectors in the basis are linearly independent, so that we can expand them.

Now we need to solve for the linear coefficients a_1, a_2, a_3 such that:

$$\mathbf{v} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$$

Substitute the actual vectors and write the equations component by component:

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = a_1\begin{pmatrix}1\\1\\0\end{pmatrix} + a_2\begin{pmatrix}1\\0\\1\end{pmatrix} + a_3\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{cases} 1 = a_1 + a_2 + a_3 & (Eq.1) \\ 2 = a_1 + a_3 & (Eq.2) \\ 3 = a_2 + a_3 & (Eq.3) \end{cases}$$

Solve for the coefficients:

(3) into (1) → $1 = a_1 + 3 \rightarrow a_1 = -2$

Into (2) → $a_3 = 2 - a_3 = 4$

Into (3) → $a_2 = 3 - a_3 = -1$

Therefore, $\mathbf{v} = \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$ in basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$

OTHER VECTOR SPACES

8) Consider the set of all complex numbers $z \in \mathbb{C}$. Is this set a vector space?

You can check that the complex numbers fulfil all the axioms.
Therefore, the set of complex numbers can be considered a vector space.

9) If the set of all complex numbers $z \in \mathbb{C}$ is indeed a vector space, what is its dimension?

The dimension of a vector space is given by the number of independent "vectors" required to span it (i.e. by the size of its basis). The problem is, how do we define the span? $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_N\} = \sum a_i \mathbf{v}_i$ for any scalar a_i. But do we allow a_i to be complex?

If we don't allow a_i to be complex, only real, then the complex numbers have two (real) dimensions. To show this, a basis for the complex numbers would be $\{\mathbf{e_1}, \mathbf{e_2}\}$ with $\mathbf{e_1} = 1$ and $\mathbf{e_2} = i$. To be a valid basis, the vectors need to be (i) linearly independent, and (ii) span the whole space

(i) Is fulfilled, because $a_1\mathbf{e_1} + a_2\mathbf{e_2} = \mathbf{0} \rightarrow a_1 + ia_2 = 0 \rightarrow a_1 = a_2 = 0$ has only the trivial solution, **if only real coefficients are allowed**.

(ii) Is true, because any complex number can be written as a linear combination of {1,i}

If on the other hand we allow a_i to be complex, then $\mathbb{C}$ has **only one** (**complex**) **dimension**, because we can define a basis $\{\mathbf{e_1} = 1\}$ and any complex vector is then written as $z = (a_1)(1) = a_1$.

INTERESTING STORY: William Rowan Hamilton, born 1805, knew that the complex numbers could be interpreted as points in a plane, and he was looking for a way to do the same for points in three-dimensional space. Points in space can be represented by their coordinates, which are triples of numbers, and for many years he had known how to add and subtract triples of numbers. However, Hamilton had been stuck on the problem of multiplication and division for a long time. Complex numbers, i.e. points in a plane, CAN be multiplied and divided with each other. However, he spent his life trying to find some system of numbers in which triplets of numbers could be multiplied and divided.

One day, crossing a bridge, he realised that what he was looking for didn't exist. The only way to have a system with numbers that can be multiplied and divided, was to have quadruplets (i.e. sets of 4 numbers, or points on a 4 dimensional space). He thus invented **quaternions**. This involved the real line (1), and three imaginary units (i, j ,k) defined by the relations $i^2 = j^2 = k^2 = ijk = -1$. Vectors in 3D space could then be represented as "pure quaternions", numbers with only imaginary components and zero real part. This gives rise to our common notation of (i, j ,k) as unit vectors in 3D space!

Importantly, he found that these quaternions **CAN** be multiplied or divided directly to produce a new quaternion. Interestingly, when multiplying two pure quaternions representing vectors in 3D space, the result is a non-pure quaternion whose real part is the **DOT PRODUCT** of the original vectors, and whose imaginary part (i,j,k) corresponds to the **CROSS PRODUCT** 3-D vector of the original numbers! **THIS IS HOW dot and cross products were "discovered"**! Later, quaternions fell largely into disuse, since it was easier to work with vectors and just define the dot and cross products axiomatically. The axioms of a vector space do not need to include the possibility of multiplying them or dividing them.

HERMITIAN INNER PRODUCT - PROOFS

10) Prove the Schwarz's inequality for complex vectors, which states:

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

Proof:

(Do not get confused between the absolute value of a scalar $|\lambda|$ and the norm of a vector $\|\mathbf{v}\|$).

Consider the linear combination $\mathbf{a} + \lambda \mathbf{b}$ and calculate its squared norm, which must always be bigger or equal to zero:

$$\|\mathbf{a} + \lambda \mathbf{b}\|^2 = \langle \mathbf{a} + \lambda \mathbf{b}, \mathbf{a} + \lambda \mathbf{b} \rangle \geq 0$$

Apply linearity in first argument of the inner product (inner product axiom):

$$\|\mathbf{a} + \lambda \mathbf{b}\|^2 = \langle \mathbf{a}, \mathbf{a} + \lambda \mathbf{b} \rangle + \lambda \langle \mathbf{b}, \mathbf{a} + \lambda \mathbf{b} \rangle \geq 0$$

Apply conjugate linearity in the second argument of the inner product (Hermitian inner product axiom):

$$\|\mathbf{a} + \lambda \mathbf{b}\|^2 = \langle \mathbf{a}, \mathbf{a} \rangle + \lambda^* \langle \mathbf{a}, \mathbf{b} \rangle + \lambda \langle \mathbf{b}, \mathbf{a} \rangle + \lambda \lambda^* \langle \mathbf{b}, \mathbf{b} \rangle \geq 0$$

Apply the conjugate symmetry of the inner product:

$$\|\mathbf{a} + \lambda \mathbf{b}\|^2 = \langle \mathbf{a}, \mathbf{a} \rangle + \lambda^* \langle \mathbf{a}, \mathbf{b} \rangle + \lambda \langle \mathbf{a}, \mathbf{b} \rangle^* + \lambda \lambda^* \langle \mathbf{b}, \mathbf{b} \rangle \geq 0$$

Now, remember that $\langle \mathbf{a}, \mathbf{a} \rangle = \|\mathbf{a}\|^2 > 0$ and $\langle \mathbf{b}, \mathbf{b} \rangle = \|\mathbf{b}\|^2 > 0$ are real positive numbers. However, do not forget that $\langle \mathbf{a}, \mathbf{b} \rangle$ **is in general a complex number** δ which has a certain amplitude and phase $\delta = |\delta| e^{i\alpha} = |\langle \mathbf{a}, \mathbf{b} \rangle| e^{i\alpha}$. Also remember the identity for complex numbers $\lambda \lambda^* = |\lambda|^2$. Substituting these we have:

$$\|\mathbf{a} + \lambda \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \lambda^* |\langle \mathbf{a}, \mathbf{b} \rangle| e^{i\alpha} + \lambda |\langle \mathbf{a}, \mathbf{b} \rangle| e^{-i\alpha} + |\lambda|^2 \|\mathbf{b}\|^2 \geq 0$$
$$\|\mathbf{a} + \lambda \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + (\lambda^* e^{i\alpha} + \lambda e^{-i\alpha}) |\langle \mathbf{a}, \mathbf{b} \rangle| + |\lambda|^2 \|\mathbf{b}\|^2 \geq 0$$

This inequality must be true for EVERY complex value of λ. To simplify the expression, let's give λ the same phase as $\langle \mathbf{a}, \mathbf{b} \rangle$ but an arbitrary modulus r, i.e. we choose $\lambda = re^{i\alpha}$:

$$\|\mathbf{a}\|^2 + (re^{-i\alpha} e^{i\alpha} + re^{i\alpha} e^{-i\alpha}) |\langle \mathbf{a}, \mathbf{b} \rangle| + r^2 \|\mathbf{b}\|^2 \geq 0$$

$$\|\mathbf{a}\|^2 + 2r |\langle \mathbf{a}, \mathbf{b} \rangle| + r^2 \|\mathbf{b}\|^2 \geq 0$$

This is a quadratic equation in r which must be positive or zero for every real value of r. Therefore, the quadratic polynomial cannot have two real solutions (i.e. it cannot cross the x-axis, as then the polynomial would be negative in some region), which means that its discriminant must be smaller than or equal to zero (i.e. imaginary solutions or one single real double solution):

$$\Delta = (2|\langle \mathbf{a}, \mathbf{b} \rangle|)^2 - 4\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \leq 0$$

$$|\langle \mathbf{a}, \mathbf{b} \rangle|^2 \leq \|\mathbf{a}\|^2 \|\mathbf{b}\|^2$$

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

11) Prove the triangle inequality for vectors, which states:

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$

We want the proof to be valid for arbitrary N-dimensional vector spaces, as long as they fulfil the axioms of a vector space and those of an inner product, so it's not enough to prove it geometrically.

Hint: make use of Schwarz's inequality!

Solution:

Since both the left- and right-hand sides are positive definite, by the positive definiteness axiom, we can square both sides and maintain the inequality (one is true if and only if the other one is true):

$$\|\mathbf{a}+\mathbf{b}\|^2 \leq (\|\mathbf{a}\| + \|\mathbf{b}\|)^2$$
$$\|\mathbf{a}+\mathbf{b}\|^2 \leq \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\|$$

The left-hand side is:

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= \langle \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}\rangle \\ &= \langle \mathbf{a}, \mathbf{a}+\mathbf{b}\rangle + \langle \mathbf{b}, \mathbf{a}+\mathbf{b}\rangle \\ &= \langle \mathbf{a}, \mathbf{a}\rangle + \langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{b}, \mathbf{a}\rangle + \langle \mathbf{b}, \mathbf{b}\rangle \\ &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{b}, \mathbf{a}\rangle\end{aligned}$$

Now, $\langle \mathbf{a}, \mathbf{b}\rangle$ and $\langle \mathbf{b}, \mathbf{a}\rangle$ are complex numbers, but thanks to the conjugate symmetry axiom we know that $\langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{b}, \mathbf{a}\rangle = \langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{a}, \mathbf{b}\rangle^*$, and we know that $z + z^* = 2\,\mathrm{Re}\{z\}$ for any complex number, so we need to prove:

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\,\mathrm{Re}\{\langle \mathbf{a}, \mathbf{b}\rangle\} \leq \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\|$$
$$\mathrm{Re}\{\langle \mathbf{a}, \mathbf{b}\rangle\} \leq \|\mathbf{a}\|\|\mathbf{b}\|$$

But the real part of any complex number is smaller than or equal to its magnitude, which fulfills $|\langle \mathbf{a}, \mathbf{b}\rangle| \leq \|\mathbf{a}\|\|\mathbf{b}\|$ by Schwartz's inequality; therefore:

$$\mathrm{Re}\{\langle \mathbf{a}, \mathbf{b}\rangle\} \leq |\langle \mathbf{a}, \mathbf{b}\rangle| \leq \|\mathbf{a}\|\|\mathbf{b}\|$$

BASIS

12) Prove that the components of $\mathbf{v}$ on a basis $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_N}\}$ are given by $v_i = \langle \mathbf{v}, \mathbf{e}_i\rangle / \langle \mathbf{e}_i, \mathbf{e}_i\rangle$ only if the basis is orthogonal.

Solution: When $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ forms a basis, then we can write $\mathbf{v} = \sum_n v_n \mathbf{e}_n$ where v_n are the components of the vector in that basis.

If we take the inner product of $\mathbf{v}$ with $\mathbf{e}_i$ then we have $\langle \mathbf{v}, \mathbf{e}_i\rangle = \langle \sum_n a_n \mathbf{e}_n, \mathbf{e}_i\rangle$ which, thanks to the linearity axiom of the inner product, can be split into a sum of inner products $\langle \mathbf{v}, \mathbf{e}_i\rangle = \sum_n a_n \langle \mathbf{e}_n, \mathbf{e}_i\rangle$.

If the basis is orthogonal, then all the terms in the sum in which $n \neq i$ will be zero, leaving only the i-th single term: $\langle \mathbf{v}, \mathbf{e}_i\rangle = v_i \langle \mathbf{e}_i, \mathbf{e}_i\rangle$, from which we can finally write $v_i = \langle \mathbf{v}, \mathbf{e}_i\rangle / \langle \mathbf{e}_i, \mathbf{e}_i\rangle$.

13) Show that $\frac{\langle \mathbf{v},\mathbf{a}\rangle}{\langle \mathbf{a},\mathbf{a}\rangle}\mathbf{a} = \langle \mathbf{v}, \hat{\mathbf{a}}\rangle\, \hat{\mathbf{a}}$ where $\hat{\mathbf{a}} = \mathbf{a}/||\mathbf{a}||$.

Solution: We apply the definition$\langle \mathbf{a}, \mathbf{a}\rangle = ||\mathbf{a}||^2$ and conjugate linearity in the 2nd argument:

$$\frac{\langle \mathbf{v}, \mathbf{a}\rangle}{\langle \mathbf{a}, \mathbf{a}\rangle}\mathbf{a} = \frac{\langle \mathbf{v}, \mathbf{a}\rangle}{||\mathbf{a}||^2}\mathbf{a} = \langle \mathbf{v}, \frac{\mathbf{a}}{||\mathbf{a}||^*}\rangle \frac{\mathbf{a}}{||\mathbf{a}||} = \langle \mathbf{v}, \frac{\mathbf{a}}{||\mathbf{a}||}\rangle \frac{\mathbf{a}}{||\mathbf{a}||} = \langle \mathbf{v}, \hat{\mathbf{a}}\rangle\, \hat{\mathbf{a}}$$

14) Determine whether the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is orthogonal, orthonormal, or otherwise.

$\mathbf{e_1} = (1,0,-1)$, $\mathbf{e_2} = (1,1,1)$, $\mathbf{e_3} = (1,-2,1)$.

Solution: We need to check the inner product for all possible pairs:

$\langle \mathbf{e_1}, \mathbf{e_2}\rangle = (1,0,-1)\cdot(1,1,1)^* = 1\cdot 1 + (-1)\cdot 1 = 0$

$\langle \mathbf{e_1}, \mathbf{e_3}\rangle = (1,0,-1)\cdot(1,-2,1)^* = 1 - 1 = 0$

$\langle \mathbf{e_2}, \mathbf{e_3}\rangle = (1,1,1)\cdot(1,-2,1)^* = 1 - 2 + 1 = 0$

So the basis is, at least, orthogonal. Is it orthonormal?

$\langle \mathbf{e_1}, \mathbf{e_1}\rangle = (1,0,-1)\cdot(1,0,-1)^* = 2$

No it is not. We could easily convert it into an orthonormal basis by normalizing each basis vector.

15) Write the vector $\mathbf{v} = (1,-3,2)$ in terms of the orthogonal basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ given by $\mathbf{e_1} = (1,0,-1)$, $\mathbf{e_2} = (1,1,1)$, $\mathbf{e_3} = (1,-2,1)$.

Solution: Normally, finding the components involves solving a linear system of equations

$$\mathbf{v} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$$

As was done in some of the problems above. Now, thanks to the basis being orthogonal, the recipe is MUCH easier!

$$a_1 = \frac{\langle \mathbf{v}, \mathbf{e_1}\rangle}{\langle \mathbf{e_1}, \mathbf{e_1}\rangle} = \frac{1-2}{1+1} = -\frac{1}{2}$$

$$a_2 = \frac{\langle \mathbf{v}, \mathbf{e_2}\rangle}{\langle \mathbf{e_2}, \mathbf{e_2}\rangle} = \frac{1-3+2}{1+1+1} = 0$$

$$a_3 = \frac{\langle \mathbf{v}, \mathbf{e_3}\rangle}{\langle \mathbf{e_3}, \mathbf{e_3}\rangle} = \frac{1+6+2}{1+4+1} = \frac{9}{6} = \frac{3}{2}$$

Indeed, you can check:

$$\mathbf{v} = \left(-\frac{1}{2}\right)\mathbf{e_1} + \frac{3}{2}\mathbf{e_3}$$

COMPLEX BASIS:

16) Determine whether the basis $\{\mathbf{e_1}, \mathbf{e_2}\} = \{\hat{\mathbf{x}} + i\hat{\mathbf{y}}, \hat{\mathbf{x}} - i\hat{\mathbf{y}}\}$ is orthogonal, orthonormal, or otherwise, using the Hermitian inner product.

Solution: To check for orthogonality, we need to check the inner product for the possible pairs, in this 2D case only one check. Remember the calculation of the inner product in terms of the components of an orthonormal basis: $\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1^* + a_2 b_2^* = \mathbf{a} \cdot \mathbf{b}^*$.

$\langle \mathbf{e_1}, \mathbf{e_2} \rangle = \langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix} \rangle = 1 \cdot 1^* + (i) \cdot (-i)^* = 1 + i^2 = 0$

So the basis is, at least, orthogonal. Is it orthonormal (i.e. unit length)?

$\langle \mathbf{e_1}, \mathbf{e_1} \rangle = \langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \rangle = 1 \cdot 1^* + (i) \cdot (i)^* = 1 \cdot 1 + (i) \cdot (-i) = 1 + 1 = 2$

No it is not. We could easily convert it into an orthonormal basis by normalizing each basis vector by $\sqrt{2}$.

17) Determine whether the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \left\{\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}}, \frac{i\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}, \frac{2+i}{\sqrt{3}}\hat{\mathbf{z}}\right\}$ is orthogonal, orthonormal, or otherwise, using the Hermitian inner product.

Solution: To check for orthogonality, we need to check the inner product for the possible pairs. Remember the calculation of the inner product in terms of the components of an orthonormal basis: $\langle \mathbf{a}, \mathbf{b} \rangle = a_x b_x^* + a_y b_y^* + a_z b_z^* = \mathbf{a} \cdot \mathbf{b}^*$

$$\langle \mathbf{e_1}, \mathbf{e_2} \rangle = \left(\frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}\right) \cdot \left(\frac{i\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}\right)^* = \left(\frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}\right) \cdot \left(\frac{-i\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}} \cdot \frac{-i}{\sqrt{2}}\right) + \left(\frac{i}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right) = 0$$

$$\langle \mathbf{e_1}, \mathbf{e_3} \rangle = \left(\frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}\right) \cdot \left(\frac{2+i}{\sqrt{3}}\hat{\mathbf{z}}\right)^* = \left(\frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}\right) \cdot \left(\frac{2-i}{\sqrt{3}}\hat{\mathbf{z}}\right) = 0$$

$$\langle \mathbf{e_2}, \mathbf{e_3} \rangle = \left(\frac{i\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}\right) \cdot \left(\frac{2+i}{\sqrt{3}}\hat{\mathbf{z}}\right)^* = \left(\frac{i\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}\right) \cdot \left(\frac{2-i}{\sqrt{3}}\hat{\mathbf{z}}\right) = 0$$

So the basis is, at least, orthogonal. Is it orthonormal?

$$\langle \mathbf{e_1}, \mathbf{e_1} \rangle = \langle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \rangle = \frac{1}{2}\langle \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \rangle = \frac{1}{2}(1 + (i)(i)^*) = 1$$

$$\langle \mathbf{e_2}, \mathbf{e_2} \rangle = \langle \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \rangle = \frac{1}{2}\langle \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \rangle = \frac{1}{2}\big((i)(i)^* + 1\big) = 1$$

$$\langle \mathbf{e_3}, \mathbf{e_3} \rangle = \langle \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 0 \\ 2+i \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 0 \\ 2+i \end{pmatrix} \rangle = \frac{1}{3}\langle \begin{pmatrix} 0 \\ 0 \\ 2+i \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2+i \end{pmatrix} \rangle = \frac{1}{3}[(2+i)(2+i)^*] = \frac{5}{3} \neq 1$$

No, $\|\mathbf{e_3}\| = \sqrt{\langle \mathbf{e_3}, \mathbf{e_3} \rangle} = \sqrt{5/3}$, so $\mathbf{e_3}$ is not a unit vector. This is an orthogonal basis, but not orthonormal. (Note: It would have been orthonormal if $\mathbf{e}_3 = \frac{2+i}{\sqrt{5}}\hat{\mathbf{z}}$).

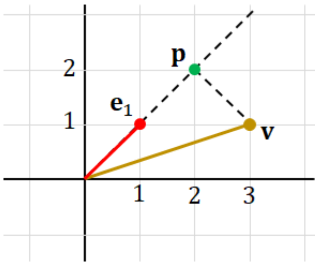
PROJECTIONS:

18) Find the projection of the vector $\mathbf{v} = (3,1)$ into the subspace defined by vector $\mathbf{e_1} = (1,1)$.

Sol:

The projection $\mathbf{p}$ of the vector $\mathbf{v}$ into the subspace, is given by a simple recipe.

$$\mathbf{p} = \frac{\langle \mathbf{v}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle}\mathbf{e_1} = \frac{\langle \binom{3}{1}, \binom{1}{1} \rangle}{\langle \binom{1}{1}, \binom{1}{1} \rangle}\binom{1}{1} = \frac{3+1}{1+1}\binom{1}{1} = \frac{4}{2}\binom{1}{1} = \binom{2}{2}$$



19) Find the projection of the vector $\mathbf{v} = (0, a, 1)$, where a is a real number, into the subspace defined by vectors $\mathbf{e_1} = (1,1,-1)$ and $\mathbf{e_2} = (1,0,1)$.

Sol: The problem is very easy once we realise that the two vectors of the subspace are orthogonal $\langle \mathbf{e_1}, \mathbf{e_2} \rangle = 1 - 1 = 0$ and, therefore, the projection $\mathbf{p}$ of the vector $\mathbf{v}$ into the subspace, is given by a simple recipe, identical to the one for finding the components of a vector in an orthogonal basis:

$$\mathbf{p} = \frac{\langle \mathbf{v}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle}\mathbf{e_1} + \frac{\langle \mathbf{v}, \mathbf{e_2} \rangle}{\langle \mathbf{e_2}, \mathbf{e_2} \rangle}\mathbf{e_2}$$
$$= \frac{a-1}{1+1+1}\mathbf{e_1} + \frac{1}{1+1}\mathbf{e_2} = \frac{a-1}{3}\mathbf{e_1} + \frac{1}{2}\mathbf{e_2}$$
$$= \frac{a-1}{3}(1,1,-1) + \frac{1}{2}(1,0,1) = \frac{1}{6}(1+2a, 2a-2, 5-2a)$$

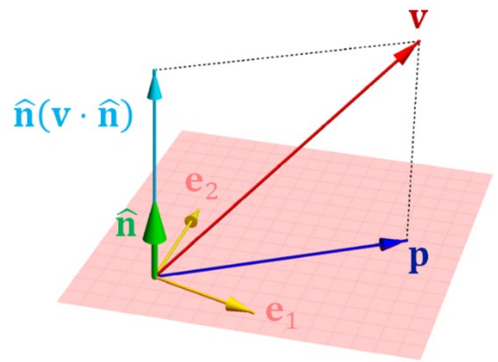
An **alternative 2 (easy method)** would be to get the original vector $\mathbf{v}$**,** and subtract from it the component that is perpendicular to the plane:

$$\mathbf{p} = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{n} \rangle}{\langle \mathbf{n}, \mathbf{n} \rangle}\mathbf{n}$$

or in more typical "vector" notation:

$$\mathbf{p} = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$

with $\hat{\mathbf{n}} = (\mathbf{e_1} \times \mathbf{e_2})/\|\mathbf{e_1} \times \mathbf{e_2}\|$



Alternative 3 (long method): This question could have been framed in the context of distances between points and planes. Find the vector within the plane given by $\mathbf{r} = \lambda(1,1,-1) + \mu(1,0,1)$ which has minimum distance to the point $\mathbf{v} = (0, a, 1)$.

In the context of distances of points and planes, this problem would be a long one: first obtain the minimum distance $d_{\min} = (\mathbf{v} - \mathbf{r_{plane}}) \cdot \hat{\mathbf{n}}$ between the plane and the vector $\mathbf{v}$ (which requires calculating $\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|}$ with $\mathbf{n} = \mathbf{e_1} \times \mathbf{e_2}$). Secondly equate $\|\mathbf{r_{plane}}(\lambda, \mu) - \mathbf{v}\| = d_{\min}$ and solve a quadratic equation in λ **and** μ (at the same time!) to find the point in the plane closest to $\mathbf{v}$. This is simplified with the knowledge that the quadratic must have only one solution, and so its discriminant must be zero.

Lookout for this in an exam: don't take the long route! Be open to different methods.

Let's do the long route just to show how inconvenient it would be:
First calculate the minimum distance to the plane:

$$\mathbf{n} = \mathbf{e_1} \times \mathbf{e_2} = (1,1,-1) \times (1,0,1) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \hat{\mathbf{x}}(1) - \hat{\mathbf{y}}(1+1) + \hat{\mathbf{z}}(-1) = (1,-2,-1)$$

$$\hat{\mathbf{n}} = \frac{\mathbf{e_1} \times \mathbf{e_2}}{\|\mathbf{e_1} \times \mathbf{e_2}\|} = \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{(1,-2,-1)}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}(1,-2,-1)$$

$$d_{\min} = (\mathbf{r_{plane}} - \mathbf{v}) \cdot \hat{\mathbf{n}} = \big((0,0,0) - (0,\mathrm{a},1)\big) \cdot \frac{1}{\sqrt{6}}(1,-2,-1) = \frac{1}{\sqrt{6}}(2a+1)$$

Second find the point of minimum distance. For a plane it's a bit more involved than for a line, because we have two free parameters.

$\|\mathbf{r_{plane}}(\lambda, \mu) - \mathbf{v}\| = d_{\min}$

$\rightarrow \|\lambda(1,1,-1) + \mu(1,0,1) - (0,a,1)\| = d_{\min}$

$\rightarrow \|(\lambda+\mu, \lambda - a, -\lambda + \mu - 1)\| = \dfrac{2a+1}{\sqrt{6}}$

Computing the norm and squaring both sides:

$\rightarrow (\lambda+\mu)^2 + (\lambda - a)^2 + (-\lambda+\mu-1)^2 = \dfrac{(2a+1)^2}{6}$

Which we can write as a quadratic equation in λ (we could also have gone for a quadratic in μ)

$\rightarrow 3\lambda^2 + (2-2a)\lambda + \left(1 + a^2 + 2\mu^2 - 2\mu - \frac{(2a+1)^2}{6}\right) = 0$ (Eq. 1)

Whose solution is given by the quadratic equation formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4a'c}}{2a'}$$

The key to solving this equation easily is to know that the solution must necessarily be a single point, and therefore the two quadratic solutions must be the same one, which means that the part inside the square root (the discriminant) must be zero:

$$b^2 - 4a'c = 0 \rightarrow (2-2a)^2 - 4(3)\left(1 + a^2 + 2\mu^2 - 2\mu - \frac{(2a+1)^2}{6}\right) = 0$$
$$\rightarrow 4 - 8a + 4a^2 - 12 - 12a^2 + 24\mu^2 - 24\mu + 8a^2 + 8a + 2 = 0$$
$$\rightarrow 6(-1 - 4\mu + 4\mu^2) = 0$$

Which, as it should be, has only one solution for μ

$$\mu = \frac{-b \pm \sqrt{b^2 - 4a'c}}{2a} = \frac{4 \pm \sqrt{16-16}}{8} = \frac{1}{2}$$

And therefore, the equation (Eq. 1) for the distance from the plane to the point when $\mu = \frac{1}{2}$ becomes:

$$3\lambda^2 + (2-2a)\lambda + \frac{(a-1)^2}{3} = 0$$

Which is a quadratic equation with only one solution for λ (as expected since we forced the determinant to be zero)

$$\lambda = \frac{2a-2}{6}$$

Substituting λ and μ into the equation of the plane, we finally find:

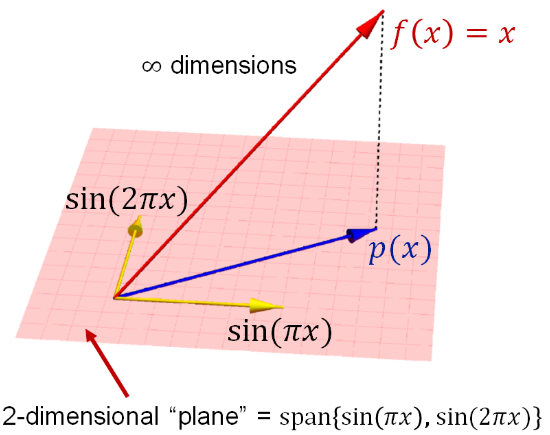
$$\mathbf{p} = \lambda(1,1,-1) + \mu(1,0,1) = \frac{2a-2}{6}(1,1,-1) + \frac{1}{2}(1,0,1) = \frac{1}{6}(1+2a, 2a-2, 5-2a)$$

Exactly as we had obtained with the inner product method in just three lines!

FUNCTION VECTOR SPACES

20) Consider the function space of functions $f(x)$ on the interval $[0,1]$. Find the projection of the function $f(x) = x$ in the subspace spanned by the functions $\{e_1(x), e_2(x), \ldots, e_M(x)\} = \{\sin(\pi x), \sin(2\pi x), \ldots, \sin(M\pi x)\}$.

Solution: Let's try to imagine what we are trying to do: consider the simple case $M = 2$. We could visualize the situation as follows:



We are projecting the infinite-dimensional vector corresponding to $f(x) = x$ with $x \in [0,1]$ into the plane formed by $\mathrm{span}\{\sin(x), \sin(2x)\}$. This plane is the "functions space" given by all possible functions that can be written as a superposition of those two sine functions:

$$\Pi(x) = a_1 \sin(\pi x) + a_2 \sin(2\pi x)$$

So, our task is to find a function which lives in that plane $p(x) \in \Pi(x)$ such that its distance to $f(x) = x$ is minimal, i.e. find the coefficients a_1 and a_2 which make the function $p(x)$ as similar as possible to $f(x) = x$.

Amazingly, the maths are identical to the ones we used with geometrical vectors! If the vectors $\mathbf{e}_i$ in the plane are orthogonal, then:

$$\mathbf{p} = \frac{\langle \mathbf{v}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle} \mathbf{e_1} + \frac{\langle \mathbf{v}, \mathbf{e_2} \rangle}{\langle \mathbf{e_2}, \mathbf{e_2} \rangle} \mathbf{e_2} + \cdots$$

So let's do it!

First, let's check if the functions $e_i(x)$ are all orthogonal to each other so that we can apply the above recipe. We could check all possible pairs separately, but in this case, we can do it in the general case:

$$\langle \mathbf{e}_m, \mathbf{e}_n \rangle = \int_0^1 g_m(x) g_n^*(x) \mathrm{d}x = \int_0^1 \sin(m\pi x) \sin(n\pi x)\, \mathrm{d}x$$

Remember $\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$. Which makes the integral easy to do:

$$\langle \mathbf{e}_m, \mathbf{e}_n \rangle = \frac{1}{2} \int_0^1 (\cos((m-n)\pi x) - \cos((m+n)\pi x)\, \mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{\sin\big((m-n)\pi x\big)}{(m-n)\pi} - \frac{\sin\big((m+n)\pi x\big)}{(m+n)\pi}\right]_{x=0}^{x=1} = \frac{\sin\big((m-n)\pi\big)}{2(m-n)\pi} - \frac{\sin\big((m+n)\pi\big)}{2(m+n)\pi}$$

We know that m and n are positive integers > 0. Therefore, the second term is always zero, while the first term is zero if $m \neq n$ and is equal to $\lim_{x\to 0} \frac{\sin(x)}{2x} = \frac{1}{2}$ when $m = n$. Indeed, we know that $\int_0^1 \sin^2(m\pi x)\, \mathrm{d}x = \frac{1}{2}$. So, in summary:

$$\langle \mathbf{e}_m, \mathbf{e}_n \rangle = \frac{1}{2}\delta_{mn}$$

So, the set of functions $\{\sin(\pi x), \sin(2\pi x), \dots, \sin(M\pi x)\}$ is an orthogonal basis!! and the norm of all the vectors in the basis is $1/\sqrt{2}$.

Now we can do the projection of $f(x) = x$ into the M-dimensional "plane" spanned by the $\mathbf{e}_i$'s:

$$\mathbf{p} = \underbrace{\frac{\langle \mathbf{f}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle}}_{a_1} \mathbf{e_1} + \underbrace{\frac{\langle \mathbf{f}, \mathbf{e_2} \rangle}{\langle \mathbf{e_2}, \mathbf{e_2} \rangle}}_{a_2} \mathbf{e_2} + \dots + \underbrace{\frac{\langle \mathbf{f}, \mathbf{e}_M \rangle}{\langle \mathbf{e}_M, \mathbf{e}_M \rangle}}_{a_M} \mathbf{e}_M$$

$$p(x) = \underbrace{\frac{\langle \mathbf{f}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle}}_{a_1} \sin(\pi x) + \underbrace{\frac{\langle \mathbf{f}, \mathbf{e_2} \rangle}{\langle \mathbf{e_2}, \mathbf{e_2} \rangle}}_{a_2} \sin(\pi x) + \dots + \underbrace{\frac{\langle \mathbf{f}, \mathbf{e}_M \rangle}{\langle \mathbf{e}_M, \mathbf{e}_M \rangle}}_{a_M} \sin(m\pi x)$$

$$p(x) = a_1 \sin(\pi x) + a_2 \sin(\pi x) + \dots + a_M \sin(m\pi x)$$

So, let's do the calculation for the coefficients a_i. We already have the bottom inner products, so let's calculate the top inner products. We can calculate them all simultaneously:

$$\langle \mathbf{f}, \mathbf{e}_m \rangle = \int_0^1 f(x) g_m^*(x) \mathrm{d}x = \int_0^1 x \sin(m\pi x)\, \mathrm{d}x$$

We can do integration by parts: $\int u\, \mathrm{d}v = u\, v - \int v\, \mathrm{d}u$ with $u = x$, $\mathrm{d}u = \mathrm{d}x$, $\mathrm{d}v = \sin(m\pi x)\mathrm{d}x$, $v = \int\, \sin(m\pi x)\mathrm{d}x = -\left(\frac{1}{m\pi}\right)\cos(m\pi x)$. So:

$$\langle \mathbf{f}, \mathbf{e}_m \rangle = \int_0^1 x \sin(m\pi x)\, \mathrm{d}x = \underbrace{\left[-\left(\frac{x}{m\pi}\right)\cos(m\pi x)\right]_{x=0}^{x=1}}_{\left(-\frac{(-1)^m}{m\pi}\right) - 0} - \frac{1}{m\pi}\underbrace{\int_0^1 \cos(m\pi x)\, \mathrm{d}x}_{\left[\frac{1}{m\pi}\sin(m\pi x)\right]_0^1 = 0}$$

$$\langle \mathbf{f}, \mathbf{e}_m \rangle = \frac{(-1)^{m-1}}{m\pi}$$

So, we have:

$$a_m = \frac{\langle \mathbf{f}, \mathbf{e}_m \rangle}{\langle \mathbf{e}_m, \mathbf{e}_m \rangle} = 2\frac{(-1)^{m-1}}{\pi m}$$

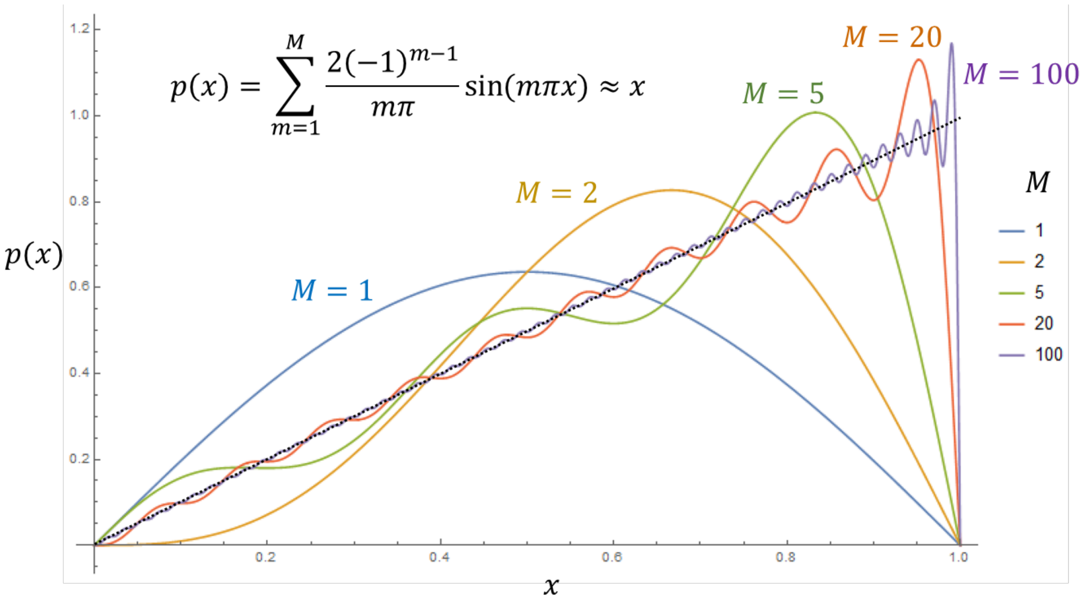
and the projection is given by:

$$p(x) = a_1 \sin(\pi x) + a_2 \sin(\pi x) + \dots + a_M \sin(m\pi x)$$

$$p(x) = \frac{2}{\pi}\sin(\pi x) + \frac{-2}{2\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \dots + \frac{2(-1)^{M-1}}{M\pi}\sin(M\pi x)$$

$$p(x) = \sum_{m=1}^{M} \frac{2(-1)^{m-1}}{m\pi}\sin(m\pi x)$$

Indeed, this projection is an excellent approximation to $f(x) = x$, and in fact gets better and better with increasing number of terms!



If you understand all the steps, you are ready to understand the Fourier series and the Fourier transform, which is a topic for second year.

2. MATRICES

2.1 INTRODUCTION TO MATRICES AS LINEAR TRANSFORMATIONS & BASIC MATRIX OPERATIONS

Pre-requisite for these notes:

Video lecture introduction to matrices and matrix-vector multiplication:
[MATRICES 1] What IS a MATRIX? Essence and Motivation (29 min)

SUMMARY OF CHAPTER 2.1

LINEAR TRANSFORMATION OF VECTORS:

$$\mathbf{v} \longrightarrow \boxed{\mathcal{A}} \longrightarrow \mathcal{A}(\mathbf{v})$$

$\mathcal{A}$ is a linear transformation $\Leftrightarrow \mathcal{A}(\alpha\mathbf{a}+\beta\mathbf{b}) = \alpha\mathcal{A}(\mathbf{a})+\beta\mathcal{A}(\mathbf{b})$

REPRESENTATION AS A MATRIX:

$$\begin{aligned}\mathcal{A}(\mathbf{v}) &= \mathcal{A}(v_1\mathbf{e}_1+\cdots+v_N\mathbf{e}_N)\\ &= v_1\mathcal{A}(\mathbf{e}_1)+\cdots+v_N\mathcal{A}(\mathbf{e}_N)\\ &\stackrel{\text{def}}{=} \begin{pmatrix} | & & | \\ \mathcal{A}(\mathbf{e}_1) & \dots & \mathcal{A}(\mathbf{e}_N) \\ | & & | \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \mathbf{A}\mathbf{v}\end{aligned}$$

SIMPLE EXAMPLES OF LINEAR TRANSFORMATIONS

Example: Rotation and scaling (both isotropic and anisotropic) in 2D and 3D.
Example: Projection of 2D and 3D space into a line or a plane.

MATRICES ACTING ON N-DIMENSIONAL SPACES

A matrix can be any size $M \times N$, converting an N dimensional input into an M dimensional output.
Example: Derivative of polynomials is a linear transformation which converts degree N to degree N-1

Vector spaces can be many different things. A matrix could represent linear transformations to polynomials, colours in {R,G,B} space, etc.

NESTED LINEAR TRANSFORMATIONS AS MATRIX MULTIPLICATION

Example: projection followed by rotation vs. rotation followed by projection – **matrices don't commute in general: $\mathbf{AB} \neq \mathbf{BA}$**
Some matrices do commute with each other $\mathbf{AB} = \mathbf{BA}$. Example: nested rotations on same axes.

SUMMARY OF BASIC MATRIX OPERATIONS

Addition of matrices $\mathbf{A}+\mathbf{B}$: add each corresponding element (matrices must have the same size)

Multiplication of matrices $\mathbf{BA}$: apply matrix multiplication $\mathbf{B}$ to each of the columns of $\mathbf{A}$

The matrix multiplication $\mathbf{BAv}$ corresponds to the nested transformations $\mathcal{B}[\mathcal{A}(\mathbf{v})]$.
Similarly to nested functions $g[f(x)]$, first apply $\mathcal{A}$ and THEN apply $\mathcal{B}$.
Dimensions of vector spaces must match throughout the "chain".
Multiplication of matrices is not commutative $\mathbf{AB} \neq \mathbf{BA}$ (e.g. project then rotate)
The rest of the properties of addition and multiplication are identical to scalars,
but remembering that **multiplication on the left** is different to **multiplication on the right**:

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$$
$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$$
$$\lambda(\mathbf{A}+\mathbf{B}) = \lambda\mathbf{A}+\lambda\mathbf{B}$$

The identity matrix $\mathbf{I}$ is the matrix version of the scalar 1. Its diagonal is 1's, rest are 0's. It does not transform vectors. Multiplying $\mathbf{I}$ left or right leaves a matrix unchanged: $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

Transpose of a matrix $\mathbf{A}^T$: swap rows by columns
$(\lambda\mathbf{A})^T = \lambda\mathbf{A}^T$
$(\mathbf{BA})^T = \mathbf{A}^T\mathbf{B}^T$ (very careful with the order, it must be swapped!)

Hermitian conjugate of a matrix: $\mathbf{A}^\dagger = (\mathbf{A}^*)^T$ – conjugate each element **and** transpose the matrix.
(using symbol 'dagger' †)
Note that Hermitian inner product: $\langle\mathbf{u}|\mathbf{v}\rangle = \mathbf{u}^\dagger\mathbf{v}$

Powers of matrices $\mathbf{A}^n = \mathbf{AA}\cdots\mathbf{A}$: multiply a matrix times itself n times.
e.g. application in "directed graphs" and "Markov processes"

ORTHOGONAL MATRICES:

$\mathbf{A}$ is an orthogonal matrix
$\Leftrightarrow \mathbf{A}^\mathrm{T} = \mathbf{A}^{-1}$
$\Leftrightarrow$ Columns of $\mathbf{A}$ form an orthonormal (real) set of vectors
$\Leftrightarrow \langle\mathbf{x}|\mathbf{y}\rangle = \langle\mathbf{Ax}|\mathbf{Ay}\rangle$ for any real vectors $\mathbf{x}$ and $\mathbf{y}$

UNITARY MATRICES:

Generalization of orthogonal matrices to matrices which are complex.

$\mathbf{A}$ is a unitary matrix
$\Leftrightarrow \mathbf{A}^\dagger = \mathbf{A}^{-1}$
$\Leftrightarrow$ Columns of $\mathbf{A}$ form an orthonormal set of vectors
$\Leftrightarrow \langle\mathbf{x}|\mathbf{y}\rangle = \langle\mathbf{Ax}|\mathbf{Ay}\rangle$ for any vectors $\mathbf{x}$ and $\mathbf{y}$

A. LINEAR TRANSFORMATIONS OF VECTORS

A general transformation of vectors has a vector as input and produces a vector as output. In general, it maps every vector in an input N dimensional vector space, into a given vector in an output M dimensional vector space. This includes the possibility of $N = M = 1$ which is the well-known case of a function $y = f(x)$.

$$\mathbf{v} \longrightarrow \boxed{\mathcal{A}} \longrightarrow \mathcal{A}(\mathbf{v})$$

There is a special type of transformations called **LINEAR TRANSFORMATIONS** with the following property:

$$\mathcal{A} \text{ is a linear transformation} \iff \mathcal{A}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathcal{A}(\mathbf{a}) + \beta\mathcal{A}(\mathbf{b})$$

Most functions ($f(x) = x^2, \sin x, e^x, 1 + x, \ldots$) are NOT linear. The only 1D linear functions that exist can always be written as a multiplication $f(x) = ax$, where a is a scalar. When the linear transformation acts on $N > 1$ vectors, the scalar multiplication $f(x) = ax$ becomes a matrix-vector multiplication $\mathcal{A}(\mathbf{v}) = \mathbf{Av}$. Therefore, **matrices are the N-dimensional equivalent to scalar multiplication!**

Once we know that a transformation is linear, it is very easy to find a way to characterise it (obtaining the "fingerprints" of the transformation) by considering the decomposition of every input vector into its basis components and applying the linearity property:

$$\begin{aligned}\mathcal{A}(\mathbf{v}) &= \mathcal{A}(v_1\mathbf{e}_1 + \cdots + v_N\mathbf{e}_N) \\ &= v_1\mathcal{A}(\mathbf{e}_1) + \cdots + v_N\mathcal{A}(\mathbf{e}_N) \\ &\stackrel{\text{def}}{=} \begin{pmatrix} | & & | \\ \mathcal{A}(\mathbf{e}_1) & \ldots & \mathcal{A}(\mathbf{e}_N) \\ | & & | \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \mathbf{Av}\end{aligned}$$

Therefore, the linear transformation is fully characterised by a list of N vectors corresponding to how the transformation acts on each basis vector of the input space: $\mathcal{A}(\mathbf{e}_1), \mathcal{A}(\mathbf{e}_2), \cdots, \mathcal{A}(\mathbf{e}_N)$. These are the "fingerprints" of the transformation. The matrix associated with this linear transformation is simply obtained by **writing these vectors as the columns of the matrix**. **Every linear transformation of vectors can be represented by a matrix once we have chosen a certain basis for the input and output space.**

A visual trick to remember matrix-vector multiplication is to shift the vector up and write the result in the vector-shaped space left in the bottom right corner. Then each element is the dot product of the corresponding row and column of the inputs:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} a_{11}v_x + a_{12}v_y \\ a_{21}v_x + a_{22}v_y \end{pmatrix} \quad \rightsquigarrow \quad \begin{array}{cc} & \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} a_{11}v_x + a_{12}v_y \\ a_{21}v_x + a_{22}v_y \end{pmatrix} \end{array}$$

B. SIMPLE EXAMPLES OF LINEAR TRANSFORMATIONS

ROTATIONS:

1) Write down the matrix associated with a 90-degree rotation counter-clockwise in two dimensions.

Solution:

Think about where the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ end up after the transformation.

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Therefore, the associated matrix is (place the resulting $\mathcal{A}(\hat{\mathbf{x}})$ and $\mathcal{A}(\hat{\mathbf{y}})$ as the columns of the matrix):

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The transformed version of any vector is given by the matrix-times-vector multiplication:

$$\mathcal{A}(\mathbf{v}) = \mathbf{A}\mathbf{v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -v_y \\ v_x \end{pmatrix}$$

2) Write down the matrix associated with a 45-degree rotation clock-wise in two dimensions.

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore, the associated matrix is:

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

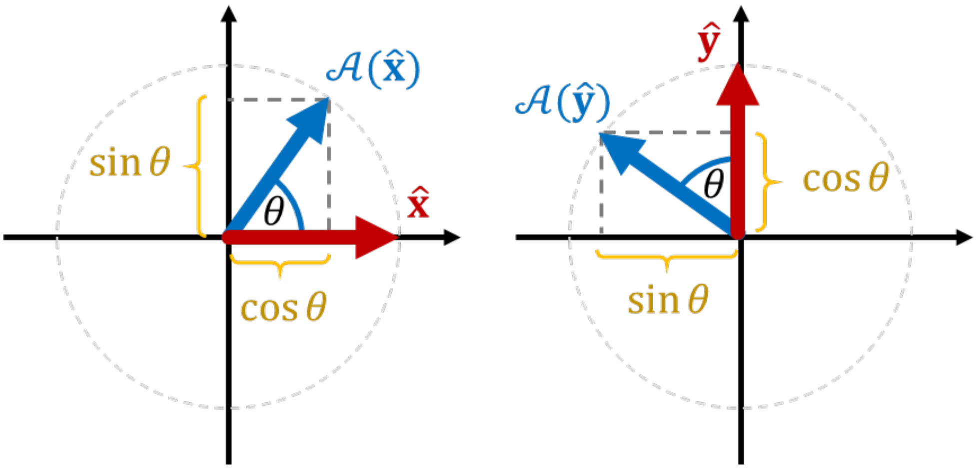
And the transformed version of any vector is:

$$\mathcal{A}(\mathbf{v}) = \mathbf{A}\mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} v_x + v_y \\ -v_x + v_y \end{pmatrix}$$

It's quite impressive that this works so easily. For example:

$\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ends up in $\mathbf{A}\mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 3+4 \\ -3+4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 7 \\ 1 \end{pmatrix}$. You would not have guessed this easily without matrices!

3) Write down the matrix associated with an arbitrary θ-degree rotation in two dimensions (as a convention, we use $\theta > 0$ for anticlockwise rotation, $\theta < 0$ for clockwise rotation, as the right hand rule results in the thumb pointing along +z or -z, respectively).



$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Therefore, the associated matrix is:

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

And the transformed version of any vector is:

$$\mathcal{A}(\mathbf{v}) = \mathbf{Av} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x\cos\theta - v_y\sin\theta \\ v_x\sin\theta + v_y\cos\theta \end{pmatrix}$$

SCALING

4) Write down the matrix associated with a scaling of 2D space by a factor K. What is the matrix associated to a transformation that leaves everything unchanged?

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \begin{pmatrix} K \\ 0 \end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \begin{pmatrix} 0 \\ K \end{pmatrix}$$

Therefore, the associated matrix is:

$$\mathbf{A} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$

And the transformed version of any vector is:

$$\mathcal{A}(\mathbf{v}) = \mathbf{Av} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} Kv_x \\ Kv_y \end{pmatrix}$$

The matrix that leaves everything unchanged is called the identity matrix, often written as $\mathbf{I}$, and corresponds to the case above when $K = 1$ (or to the 2D rotation matrix when $\theta = 0$).

5) Write down the matrix which scales space in the x-direction by a factor of 2, and scales space in the y-direction by a factor of $1/2$.

$\hat{\mathbf{x}} = \begin{pmatrix}1\\0\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \begin{pmatrix}2\\0\end{pmatrix}$

$\hat{\mathbf{y}} = \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \begin{pmatrix}0\\1/2\end{pmatrix}$

Therefore, the associated matrix is:

$\mathbf{A} = \begin{pmatrix}2 & 0\\0 & 1/2\end{pmatrix}$

PROJECTION IN 2D

6) Write down the matrix whose associated linear transformation projects 2-D space into the x-axis.

$\hat{\mathbf{x}} = \begin{pmatrix}1\\0\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \begin{pmatrix}1\\0\end{pmatrix}$

$\hat{\mathbf{y}} = \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \begin{pmatrix}0\\0\end{pmatrix}$

Therefore, the associated matrix is:

$\mathbf{A} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

And any vector lands in the x-axis:

$$\mathcal{A}(\mathbf{v}) = \mathbf{A}\mathbf{v} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}v_x\\v_y\end{pmatrix} = \begin{pmatrix}v_x\\0\end{pmatrix}$$

7) Consider the projection of vectors into the line $y = mx + c$. Is it a linear transformation? If so, write down the associated matrix.

A projection into the line $y = mx + c$ is a vector transformation, but is it a **linear** vector transformation? It will be a LINEAR transformation, and therefore can be represented by a matrix, **only if** $\mathcal{A}(\lambda\mathbf{v}) = \lambda\mathcal{A}(\mathbf{v})$ for any value of λ. This includes $\lambda = 0$ which gives $\mathcal{A}(\mathbf{0}) = \mathbf{0}$. Every linear transformation must map 0 to 0, by definition of linearity. Therefore projection into $y = mx + c$ is not a linear transformation unless the line crosses the origin: i.e., $c = 0$, because otherwise, (0,0) would not be projected onto (0,0). So we will assume $c = 0$ as a necessary condition.

We are interested on the linear transformation which projects 2-D space into the line $y = mx$. The projection $\mathbf{p} = \mathcal{A}(\mathbf{v})$, of a vector $\mathbf{v}$ into a direction given by the unit vector $\hat{\mathbf{u}}$ is given by $\mathbf{p} = \langle\mathbf{v}, \hat{\mathbf{u}}\rangle\hat{\mathbf{u}}$.

In the line $y = mx$, we know that m represents the gradient, so when x increases by 1, y increases by m. Therefore, the unit vector $\hat{\mathbf{u}}$ can be obtained as $\hat{\mathbf{u}} = \frac{(1,m)}{|(1,m)|} = \frac{1}{\sqrt{1+m^2}}\begin{pmatrix}1\\m\end{pmatrix}$.

We can now apply the projection to each unit vector:

$$\hat{\mathbf{x}} = \begin{pmatrix}1\\0\end{pmatrix} \to \mathcal{A}(\hat{\mathbf{x}}) = \langle\hat{\mathbf{x}},\hat{\mathbf{u}}\rangle\hat{\mathbf{u}} = \langle\begin{pmatrix}1\\0\end{pmatrix}, \frac{1}{\sqrt{1+m^2}}\begin{pmatrix}1\\m\end{pmatrix}\rangle \frac{1}{\sqrt{1+m^2}}\begin{pmatrix}1\\m\end{pmatrix} = \frac{1}{1+m^2}\begin{pmatrix}1\\m\end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix}0\\1\end{pmatrix} \to \mathcal{A}(\hat{\mathbf{y}}) = \langle\hat{\mathbf{y}},\hat{\mathbf{u}}\rangle\hat{\mathbf{u}} = \langle\begin{pmatrix}0\\1\end{pmatrix}, \frac{1}{\sqrt{1+m^2}}\begin{pmatrix}1\\m\end{pmatrix}\rangle \frac{1}{\sqrt{1+m^2}}\begin{pmatrix}1\\m\end{pmatrix} = \frac{m}{1+m^2}\begin{pmatrix}1\\m\end{pmatrix}$$

Therefore, the associated matrix is, as a function of m:

$$\mathbf{A}(m) = \frac{1}{1+m^2}\begin{pmatrix}1 & m\\ m & m^2\end{pmatrix}$$

And any vector lands in the $y = mx$ line:

$$\mathcal{A}(\mathbf{v}) = \mathbf{A}\mathbf{v} = \frac{1}{1+m^2}\begin{pmatrix}1 & m\\ m & m^2\end{pmatrix}\begin{pmatrix}v_x\\v_y\end{pmatrix} = \frac{1}{1+m^2}\begin{pmatrix}v_x + mv_y\\ mv_x + m^2v_y\end{pmatrix} = \frac{v_x + mv_y}{1+m^2}\hat{\mathbf{x}} + \frac{mv_x + m^2v_y}{1+m^2}\hat{\mathbf{y}}$$

Interestingly, notice that when $m = 0$, the matrix becomes $\mathbf{A}(0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ which corresponds to the projection into the x-axis.

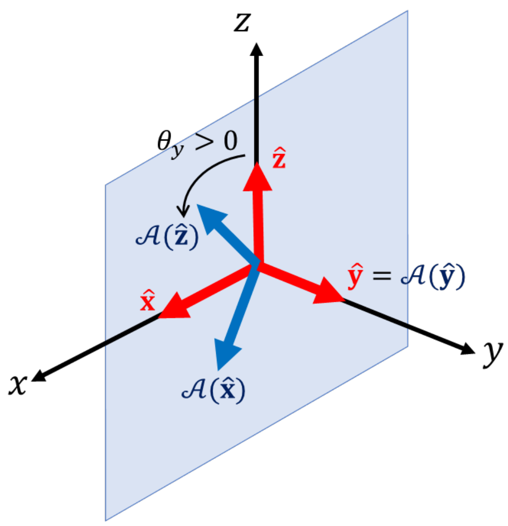
Interestingly, $\mathbf{A}(\infty) = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$ if properly performing the limit $m \to \infty$, and this is indeed the matrix which corresponds to a projection into the y-axis.

ROTATION IN 3D

8) Write down the matrix associated with an arbitrary θ_y-degree rotation around the y axis. The main difficulty in this problem is to get the signs correct (these signs are all an artificial convention but must be consistent. The convention is to use the right-hand rule to define the sign of θ_y according to whether the thumb points along $+y$ or $-y$ when the other fingers move in the direction of the rotation). **Suggestion**: draw a diagram.

Solution:

Draw the 3 unit vectors (remembering that x-y-z form a right-handed triplet such that $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$) and then draw their transformed version. Next consider a rotation with $\theta > 0$ which using the right hand rule results in the thumb pointing along the positive y direction.



Then use trigonometry to write down the transformed version of the three unit vectors:

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \begin{pmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{z}}) = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

So the rotation matrix in 3D around the y-axis is given by (this matrix is used often in different mathematical contexts and usually named as $\mathbf{R}_y$):

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

You should be able to find the rotation matrices for rotations around the other two axes.

PROJECTION IN 3D

9) Write down the matrix whose associated linear transformation projects all of 3-D space into the plane given as $\Pi = \mathrm{span}\{\mathbf{e_1}, \mathbf{e_2}\}$, with $\mathbf{e_1} = (0,1,0)^T$ and $\mathbf{e_2} = (1,0,1)^T$

The projection $\mathbf{p}$ of a vector $\mathbf{v}$ to a subspace $\mathrm{span}\{\mathbf{e_1}, \mathbf{e_2}\}$ can be written using a simple recipe if the basis of the subspace is orthogonal. In this case, $\langle \mathbf{e_1}, \mathbf{e_2} \rangle = 0$, the basis is orthogonal, so we can apply the simple recipe:

$$\mathbf{p} = \frac{\langle \mathbf{v}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle}\mathbf{e_1} + \frac{\langle \mathbf{v}, \mathbf{e_2} \rangle}{\langle \mathbf{e_2}, \mathbf{e_2} \rangle}\mathbf{e_2}$$

So, applying this projection ($\mathbf{e_1} = (0,1,0)^T$ and $\mathbf{e_2} = (1,0,1)^T$) to the unit vectors:

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \frac{\langle \hat{\mathbf{x}}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle}\mathbf{e_1} + \frac{\langle \hat{\mathbf{x}}, \mathbf{e_2} \rangle}{\langle \mathbf{e_2}, \mathbf{e_2} \rangle}\mathbf{e_2} = 0 + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \frac{\langle \hat{\mathbf{y}}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle}\mathbf{e_1} + \frac{\langle \hat{\mathbf{y}}, \mathbf{e_2} \rangle}{\langle \mathbf{e_2}, \mathbf{e_2} \rangle}\mathbf{e_2} = 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0$$

$$\hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{z}}) = \frac{\langle \hat{\mathbf{z}}, \mathbf{e_1} \rangle}{\langle \mathbf{e_1}, \mathbf{e_1} \rangle}\mathbf{e_1} + \frac{\langle \hat{\mathbf{z}}, \mathbf{e_2} \rangle}{\langle \mathbf{e_2}, \mathbf{e_2} \rangle}\mathbf{e_2} = 0 + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

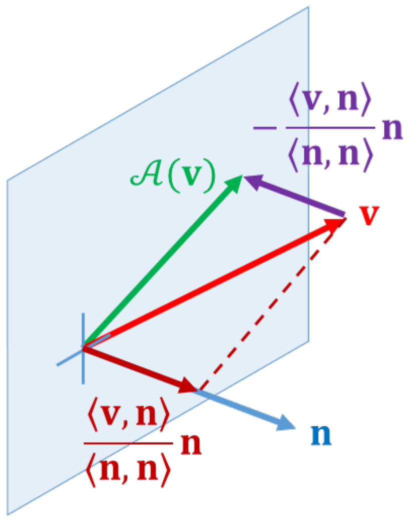
Therefore, the associated matrix is

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

A quick check that this is correct can be to test how the matrix acts on vectors $\mathbf{e_1}$ and $\mathbf{e_2}$, which, being in the plane itself, should be unchanged by the projection. Indeed $\mathbf{Ae_1} = \mathbf{e_1}$ and $\mathbf{Ae_2} = \mathbf{e_2}$.

10) Write down the matrix whose associated linear transformation projects all 3-D space into the plane given by $x+y=0$.

This time, the plane is given in terms of its normal vector in the form $n_x x + n_y y + n_z z = 0$, so we know that $\mathbf{n} = (1,1,0)$.
As we know from the previous problem, the projection of a vector $\mathbf{v}$ to a subspace $\operatorname{span}\{\mathbf{e_1}, \mathbf{e_2}\}$ can be written using a simple recipe if the basis of the subspace is orthogonal. A possible method would be therefore to **find two orthogonal vectors within the plane** and apply the method in the previous problem. However, since we have the normal vector to the plane, and we are working in 3D space, there is an easier method. See the figure:



Clearly, any vector $\mathbf{v}$ can be written as a vector within the plane (subspace) which is the projection $\mathcal{A}(\mathbf{v})$, added to a vector parallel to $\mathbf{n}$, so $\mathbf{v} = \mathcal{A}(\mathbf{v}) + \frac{\langle \mathbf{v},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n}$, where the second term parallel to $\mathbf{n}$ is simply the projection of $\mathbf{v}$ on the direction of $\mathbf{n}$. Therefore, $\mathcal{A}(\mathbf{v})$ is given by subtracting this component to $\mathbf{v}$, such that:

$$\mathcal{A}(\mathbf{v}) = \mathbf{v} - \frac{\langle \mathbf{v},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n}$$

So, applying this projection to the unit vectors:

$$\hat{\mathbf{x}} = \begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \hat{\mathbf{x}} - \frac{\langle \hat{\mathbf{x}},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1+0+0}{1+1+0}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \hat{\mathbf{y}} - \frac{\langle \hat{\mathbf{y}},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{0+1+0}{1+1+0}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\hat{\mathbf{z}} = \begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{z}}) = \hat{\mathbf{z}} - \frac{\langle \hat{\mathbf{z}},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{0+0+0}{1+1+0}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Therefore, the associated matrix is

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 2\end{pmatrix}$$

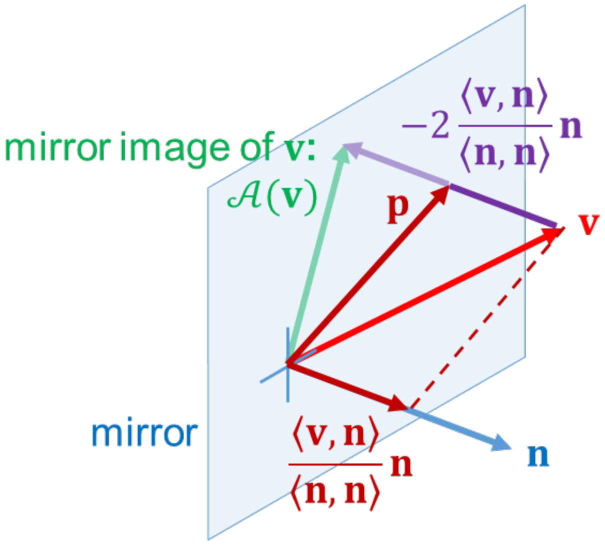
A quick check that this is correct can be to test how the matrix acts on vector $\mathbf{n}$. Indeed $\mathbf{An} = \mathbf{0}$.

MIRROR SYMMETRY IN 3D

11) Write down the matrix whose associated linear transformation performs a mirror-symmetry operation on 3-D space, with the mirror-plane given by $x + y = 0$.

The normal vector of this mirror plane ($n_x x + n_y y + n_z z = d$) is given by $\mathbf{n} = (1,1,0)$.

Mirror symmetry is a transformation which flips the sign of the component of every vector $\mathbf{v}$ along the direction normal to the mirror, $\mathbf{n}$.



That is, if $\mathbf{p}$ is the projection of a vector $\mathbf{v}$ on the mirror plane, then that component remains unchanged by the linear transformation, but the component normal to the mirror plane is flipped:

$$\mathbf{v} = \mathbf{p} + \frac{\langle \mathbf{v},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n} \;\rightarrow\; \mathcal{A}(\mathbf{v}) = \mathbf{p} - \frac{\langle \mathbf{v},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n}$$

Therefore:

$$\mathcal{A}(\mathbf{v}) = \mathbf{v} - 2\frac{\langle \mathbf{v},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n}$$

So, applying this mirror-symmetry to the unit vectors:

$$\hat{\mathbf{x}} = \begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{x}}) = \hat{\mathbf{x}} - 2\frac{\langle \hat{\mathbf{x}},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n} = \begin{pmatrix}1\\0\\0\end{pmatrix} - 2\frac{1+0+0}{1+1+0}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{y}}) = \hat{\mathbf{y}} - 2\frac{\langle \hat{\mathbf{y}},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n} = \begin{pmatrix}0\\1\\0\end{pmatrix} - 2\frac{0+1+0}{1+1+0}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$$

$$\hat{\mathbf{z}} = \begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow \mathcal{A}(\hat{\mathbf{z}}) = \hat{\mathbf{z}} - 2\frac{\langle \hat{\mathbf{z}},\mathbf{n}\rangle}{\langle \mathbf{n},\mathbf{n}\rangle}\mathbf{n} = \begin{pmatrix}0\\0\\1\end{pmatrix} - 2\frac{0+0+0}{1+1+0}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Therefore, the associated matrix is

$$\mathbf{A} = \begin{pmatrix}0 & -1 & 0\\-1 & 0 & 0\\0 & 0 & 1\end{pmatrix}$$

A quick check that this is correct can be to test how the matrix acts on vector $\mathbf{n}$. Indeed $\mathbf{An} = -\mathbf{n}$.

ANISOTROPIC SCALING IN NON-ORTHOGONAL DIRECTIONS (2D)

12) Write down the matrix whose associated linear transformation scales space in the direction $\mathbf{v}_1 = (2,1)$ by a factor of 2, and scales space in the direction $\mathbf{v}_2 = (1,2)$ by a factor of $1/2$.

Solution:

In order to find the matrix associated with this transformation, we need to find how this transformation affects each of the unit vectors.

How are we going to do this?

First, let's think how the transformation affects ANY vector $\mathbf{a} \to \mathcal{A}(\mathbf{a})$.

The transformation scales space in the direction $\mathbf{v}_1 = (2,1)$ by a factor of 2, and scales space in the direction $\mathbf{v}_2 = (1,2)$ by a factor of ½, therefore, remembering the linearity of the operation, we can write any vector as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and then scale them appropriately:

$$\mathbf{a} = a_1\mathbf{v_1} + a_2\mathbf{v_2} \xrightarrow{\mathcal{A}} \mathcal{A}(\mathbf{a}) = a_1\mathcal{A}(\mathbf{v_1}) + a_2\mathcal{A}(\mathbf{v_2}) = a_1(2\mathbf{v_1}) + a_2\left(\frac{1}{2}\mathbf{v_2}\right)$$

If we want to do this with the unit vectors, we need to write each of the unit vectors as a linear combination of the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. This is easy if vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, however $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 2 + 2 = 4 \neq 0$, so they are not orthogonal. We have to find the components in the usual way, solving a system of equations.

$$\hat{\mathbf{x}} = \begin{pmatrix}1\\0\end{pmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\begin{pmatrix}2\\1\end{pmatrix} + a_2\begin{pmatrix}1\\2\end{pmatrix}$$

Writing the equations component by component:

$$\begin{cases} 2a_1 + a_2 = 1 & (Eq.1) \\ a_1 + 2a_2 = 0 & (Eq.2) \end{cases}$$

(1)-2(2) $\to$ $-3a_2 = 1 \to a_2 = -\frac{1}{3}$

Into (2) $\to a_1 = -2a_2 = \frac{2}{3}$

$$\text{So } \hat{\mathbf{x}} = \frac{2}{3}\begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix}$$

Repeating the procedure with $\hat{\mathbf{y}}$:

$$\hat{\mathbf{y}} = \begin{pmatrix}0\\1\end{pmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\begin{pmatrix}2\\1\end{pmatrix} + a_2\begin{pmatrix}1\\2\end{pmatrix}$$

Writing the equations component by component:

$$\begin{cases} 2a_1 + a_2 = 0 & (Eq.1) \\ a_1 + 2a_2 = 1 & (Eq.2) \end{cases}$$

(1)-2(2) $\to$ $-3a_{2x} = -2 \to a_{2x} = \frac{2}{3}$

Into (2) $\to a_{1x} = 1 - 2a_{2x} = \frac{-1}{3}$

$$\text{So } \hat{\mathbf{y}} = -\frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}1\\2\end{pmatrix}$$

Once we know how to write $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in terms of those two special directions, we can apply the transformation to each component separately.

$$\hat{\mathbf{x}} = \begin{pmatrix}1\\0\end{pmatrix} = \frac{2}{3}\mathbf{v_1} - \frac{1}{3}\mathbf{v_2} \xrightarrow{\mathcal{A}} \mathcal{A}(\hat{\mathbf{x}}) = \frac{2}{3}\mathcal{A}(\mathbf{v_1}) - \frac{1}{3}\mathcal{A}(\mathbf{v_2})$$

$$= \frac{2}{3}(2\mathbf{v_1}) - \frac{1}{3}\left(\frac{1}{2}\mathbf{v_2}\right) = \frac{2}{3}2\begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{3}\frac{1}{2}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5/2\\1\end{pmatrix}$$

$$\hat{\mathbf{y}} = \begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{3}\mathbf{v_1} + \frac{2}{3}\mathbf{v_2} \xrightarrow{\mathcal{A}} \mathcal{A}(\hat{\mathbf{y}}) = -\frac{1}{3}\mathcal{A}(\mathbf{v_1}) + \frac{2}{3}\mathcal{A}(\mathbf{v_2}) = \left(-\frac{1}{3}\right)2\begin{pmatrix}2\\1\end{pmatrix} + \frac{2}{3}\frac{1}{2}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

Therefore, the associated matrix is:

$$\mathbf{A} = \begin{pmatrix}5/2 & -1\\1 & 0\end{pmatrix}$$

You can easily check that $\mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1$ and that $\mathbf{A}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_2$ as required.

$$\mathbf{A}\mathbf{v}_1 = \begin{pmatrix}5/2 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix} \quad \text{and} \quad \mathbf{A}\mathbf{v}_2 = \begin{pmatrix}5/2 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1/2\\1\end{pmatrix}$$

Demonstrate using Geogebra. Point out we would never guess those special directions by looking at the transformation alone.

C. LINEAR TRANSFORMATIONS IN N-DIMENSIONAL SPACES

Let's not limit ourselves to geometrical spaces in $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$.

Linear transformations, and therefore matrices, act on any possible (finite dimensional) vector space.

Vector spaces can be many different things. A matrix could represent a conversion from polynomials to colours in {R,G,B} space, for example.

13) Consider the vector space of polynomials of degree equal or smaller than 4, using the basis $\{1, x, x^2, x^3, x^4\}$. Answer the following:

a) Is the derivative of a polynomial $\frac{\mathrm{d}}{\mathrm{d}x}(p(x))$ a linear transformation of $p(x)$ in this vector space?

b) If so, obtain the matrix which represents the derivative.

Solution:

a) The definition of a linear transformation is the following. For any two vectors $\mathbf{u}$ and $\mathbf{v}$, and any two coefficients λ and μ, the transformation $\mathcal{A}$ fulfils:

$$\mathcal{A}(\lambda\mathbf{u} + \mu\mathbf{v}) = \lambda\mathcal{A}(\mathbf{u}) + \mu\mathcal{A}(\mathbf{v})$$

We know that polynomials can be interpreted as vectors, so exactly the same definition for linear transformation applies. A transformation of polynomials is linear if and only if:

$$\mathcal{A}[\lambda\, p(x) + \mu\, q(x)] = \lambda\mathcal{A}[p(x)] + \mu\mathcal{A}[q(x)]$$

For any two polynomials $p(x)$ and $q(x)$. Does the derivative fulfil this condition? Indeed:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\lambda\, p(x) + \mu\, q(x)] = \lambda\frac{\mathrm{d}}{\mathrm{d}x}[p(x)] + \mu\frac{\mathrm{d}}{\mathrm{d}x}[q(x)]$$

is fulfilled. Therefore, **the derivative of a polynomial IS a linear transformation** (in fact, the above equation works for any function, not only polynomials, so the derivative is a linear transformation in general)!

b) If the derivative is a linear transformation acting on a finite-dimensional space, then it necessarily has a matrix representation! To find the matrix, we need to have a clearly defined basis for the input and output space. In this case, the questions asks us to use the basis $\{1, x, x^2, x^3, x^4\}$.

To find the matrix, we need to apply the linear transformation to each of the basis elements:

$1 \to \mathcal{A}(1) = \frac{\mathrm{d}}{\mathrm{d}x}[1] = 0$ whose vector representation is $(0,0,0,0,0)^T$

$x \to \mathcal{A}(x) = \frac{\mathrm{d}}{\mathrm{d}x}[x] = 1$ whose vector representation is $(1,0,0,0,0)^T$

$x^2 \to \mathcal{A}(x^2) = \frac{\mathrm{d}}{\mathrm{d}x}[x^2] = 2x$ whose vector representation is $(0,2,0,0,0)^T$

$x^3 \to \mathcal{A}(x^3) = \frac{\mathrm{d}}{\mathrm{d}x}[x^3] = 3x^2$ whose vector representation is $(0,0,3,0,0)^T$

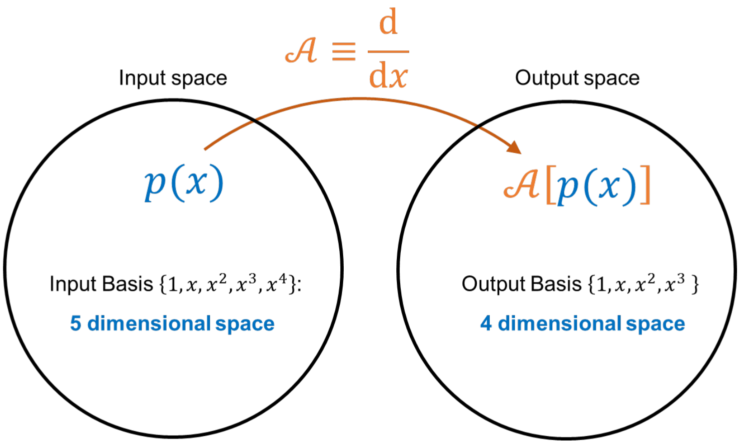
$x^4 \to \mathcal{A}(x^4) = \frac{\mathrm{d}}{\mathrm{d}x}[x^4] = 4x^3$ whose vector representation is $(0,0,0,4,0)^T$

Therefore, the associated matrix is obtained by placing the different outputs as the columns in the grid:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that the last row is all zeroes. That is because the derivative of a polynomial of degree 4 is always a polynomial of, at most, degree 3. We could therefore interpret the derivative as acting on the space of polynomials of degree 4 or less, and placing the output into the space of polynomials of degree 3 or less. This can be represented by a **non-square matrix, whose number of columns is equal to the dimension of the input space, and whose number of rows is equal to the dimension of the output space. In this case it is a 4×5 matrix.**

$$\mathbf{A}' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$



Indeed, the matrix-vector multiplication must fulfil the following conditions on the number of dimensions of the input and output vectors. The dimension of the input vector must be equal to the number of columns of the matrix (5 dimensional input). The dimensions of the output is equal to the number of rows of the matrix (4 dimensional output).

In this way, a matrix can be any size $M \times N$, converting an N dimensional input into an M dimensional output.

14) Linear RGB colorspace uses the three components (R, G, B) to determine the intensity of each coloured subpixel. However, sometimes we want an image to be monochromatic (black & white & shades of grey) and therefore each pixel should be defined by its intensity, a single value (Y).

When converting a colour image into a grayscale image, we could just project each vector (R,G,B) into the line of grayscale $\lambda(1,1,1)$, however this would ignore the fact that the different colours are perceived with different brightness. Scientists and psychologists determined that a pure blue colour is perceived as less bright than a pure green colour (for equal light intensity). This is related to biology of the eye and perception. Therefore, the RGB values should not contribute equally to the luminosity for the best result. The following international standard is given for conversion of RGB to luminosity values.

$$Y = 0.2126R + 0.7152G + 0.0722B$$

Find the matrix associated to this linear transformation.

Solution:

This linear transformation converts a three-dimensional space (RGB) into a one-dimensional space (Y). Therefore, the matrix must have 1 row and 3 columns. Each column must be the luminosity Y corresponding to each of the basis vectors RGB. The matrix is simply given as:

$$\mathbf{A} = \begin{pmatrix} 0.2126 & 0.7152 & 0.0722 \end{pmatrix}$$

So that, indeed:

$$Y = \mathbf{Ac} = \begin{pmatrix} 0.2126 & 0.7152 & 0.0722 \end{pmatrix} \begin{pmatrix} R \\ G \\ B \end{pmatrix} = 0.2126R + 0.7152G + 0.0722B$$

Notice something that is often used: An $M \times 1$ matrix is equivalent to the dot product operation for vectors of dimension M. In fact, some books denote the dot product of $\mathbf{u}$ and $\mathbf{v}$ as the matrix-vector multiplication $\mathbf{u}^T\mathbf{v}$ where, if one wishes, we can see the transposed vector $\mathbf{u}^T$ as a matrix.

D. BASIC MATRIX OPERATIONS

ADDITION OF MATRICES

Addition of matrices $\mathbf{A}+\mathbf{B}$: add each corresponding element (matrices must have same size)

This corresponds to a linear transformation $\mathcal{C}(\mathbf{v})=\mathcal{A}(\mathbf{v})+\mathcal{B}(\mathbf{v})$. This only makes sense if the input and output vector spaces have the same dimension, i.e. matrices have same size.

15) Find $\begin{pmatrix}1 & a\\ 0 & 2\end{pmatrix}+\begin{pmatrix}0 & \pi\\ a & 1\end{pmatrix}$

Solution: Simply add element by element: $\begin{pmatrix}1 & a\\ 0 & 2\end{pmatrix}+\begin{pmatrix}0 & \pi\\ a & 1\end{pmatrix}=\begin{pmatrix}1 & a+\pi\\ a & 3\end{pmatrix}$

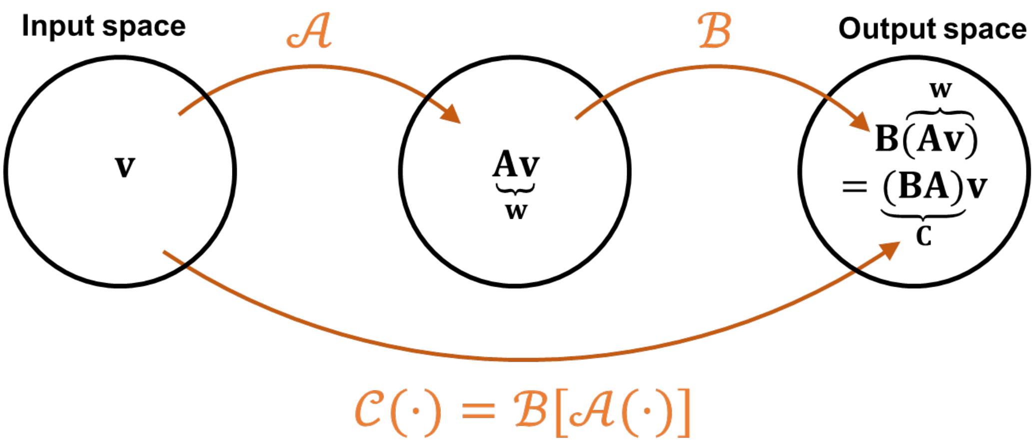
16) Find $\begin{pmatrix}1 & a\\ 0 & 2\end{pmatrix}+\begin{pmatrix}0 & 1 & \pi\\ a & 2 & 1\end{pmatrix}$

Solution: They cannot be added because they have different size

MULTIPLICATION OF MATRICES BA

The matrix multiplication $\mathbf{C}=\mathbf{BA}$ corresponds to the nested transformation $\mathcal{C}(\mathbf{v})=\mathcal{B}[\mathcal{A}(\mathbf{v})]$. Similarly to nested functions $g[f(x)]$, first apply $\mathcal{A}$ and THEN apply $\mathcal{B}$. Dimensions of vector spaces must match throughout the "chain".

- Consider a nested linear transformation $\mathcal{C}(\mathbf{v})=\mathcal{B}[\mathcal{A}(\mathbf{v})]$.



<u>1×1 **dimensional case:**</u>

Linear transformations are given by good old multiplication by a scalar:
Transformation $\mathcal{A}(x)=ax$
Transformation $\mathcal{B}(x)=bx$
Nested transformation $\mathcal{C}(x)=\mathcal{B}[\mathcal{A}(x)]=bax=cx$ with $c=ba$

So 1×1 dimensional matrices are multiplied by simply multiplying its single elements together. What happens to general-sized matrices? Things get interesting and beautiful.

<u>**General dimensional case**</u>**.** Write the input vector $\mathbf{v}$ as a combination of the input basis vectors:

$$\mathcal{B}[\mathcal{A}(\mathbf{v})] = \mathcal{B}[\mathcal{A}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_N\mathbf{e}_N)]$$

Thanks to the linearity of the transformations $\mathcal{A}$ and $\mathcal{B}$, we can rewrite this as:

$$\mathcal{B}[\mathcal{A}(\mathbf{v})] = a_1\mathcal{B}[\mathcal{A}(\mathbf{e}_1)] + a_2\mathcal{B}[\mathcal{A}(\mathbf{e}_2)] + \cdots + a_N\mathcal{B}[\mathcal{A}(\mathbf{e}_N)]$$
$$\mathcal{C}(\mathbf{v}) = a_1\mathcal{C}(\mathbf{e}_1) + a_2\mathcal{C}(\mathbf{e}_2) + \cdots + a_N\mathcal{C}(\mathbf{e}_N)$$

Therefore, the *i*-th column of $\mathbf{C}$ is given by the nested transformation applied to each of the input basis vectors:

$$\mathbf{C} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathcal{B}[\mathcal{A}(\mathbf{e}_1)] & \mathcal{B}[\mathcal{A}(\mathbf{e}_2)] & \ldots & \mathcal{B}[\mathcal{A}(\mathbf{e}_N)] \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

This matrix $\mathbf{C}$ can be defined to be the product $\mathbf{C} = \mathbf{BA}$, where $\mathbf{B}$ and $\mathbf{A}$ represent the matrices for the individual transformations.

Remember that $\mathcal{A}(\mathbf{e}_i)$ is nothing else than the columns of matrix $\mathbf{A}$.

Therefore, each column of matrix $\mathbf{C} = \mathbf{BA}$ corresponds to the matrix $\mathbf{B}$ multiplied with each of the columns of matrix $\mathbf{A}$.

Multiplication of matrices $\mathbf{BA}$: apply matrix multiplication $\mathbf{B}$ to each of the columns of $\mathbf{A}$

17) Calculate the matrix multiplication $\begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & \pi \\ a & 1 \end{pmatrix}$

Solution: Apply matrix multiplication $\begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}$ to each of the columns of $\begin{pmatrix} 0 & \pi \\ a & 1 \end{pmatrix}$

First column is $\begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} a^2 \\ 2a \end{pmatrix}$

Second column is $\begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \pi \\ 1 \end{pmatrix} = \begin{pmatrix} \pi + a \\ 2 \end{pmatrix}$

Therefore: $\begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & \pi \\ a & 1 \end{pmatrix} = \begin{pmatrix} a^2 & \pi + a \\ 2a & 2 \end{pmatrix}$

The multiplication can be done **directly** with the same trick we use for matrix-vector multiplication, placing the second matrix shifted upwards, and looking at the matrix-shaped space at the bottom right corner, where the output goes.

$$\begin{array}{cc} & \begin{pmatrix} 0 & \pi \\ a & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} a^2 & \pi + a \\ 2a & 2 \end{pmatrix} \end{array}$$

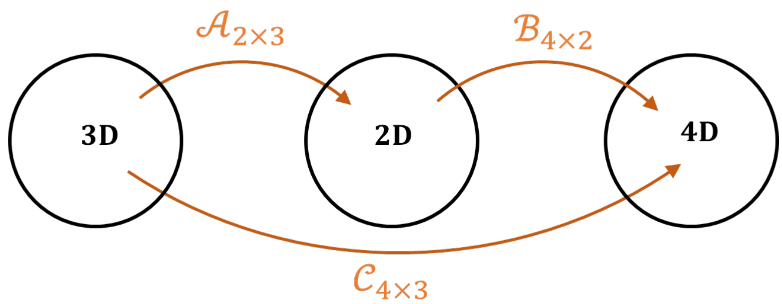
Each element of the output is the dot product of the corresponding row and column in the input matrices being multiplied. This has the added advantage of being an **in-built dimension mismatch check**.

18) Calculate the matrix multiplication $\begin{pmatrix} 1 & a \\ 0 & 2 \\ 1 & 1 \\ k & 0 \end{pmatrix}\begin{pmatrix} 0 & \pi & \sqrt{2} \\ a & 1 & 0 \end{pmatrix}$

Even though the matrices have different sizes, we can still apply the matrix $\mathbf{B}$ to each of the columns of matrix $\mathbf{A}$ (because the dimensions are compatible for matrix-vector mutiplication). Therefore, we can obtain:

$$\begin{pmatrix} 1 & a \\ 0 & 2 \\ 1 & 1 \\ k & 0 \end{pmatrix}\begin{pmatrix} 0 & \pi & \sqrt{2} \\ a & 1 & 0 \end{pmatrix} = \begin{pmatrix} a^2 & \pi + a & \sqrt{2} \\ 2a & 2 & 0 \\ a & \pi + 1 & \sqrt{2} \\ 0 & k\pi & \sqrt{2}k \end{pmatrix}$$

Think about how each matrix modifies the dimensions of its input and output spaces. Remember that an $M \times N$ matrix converts an N dimensional space into an M dimensional space. Dimensions match throughout the chain, like this:

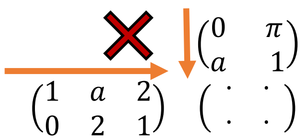


19) Calculate the matrix multiplication $\begin{pmatrix} 1 & a & 2 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & \pi \\ a & 1 \end{pmatrix}$

Solution:

This matrix multiplication cannot be done, because dimensions do not match! $\mathbf{A}$ converts 2D to 2D, while $\mathbf{B}$ converts 3D to 2D. They cannot be concatenated as $\mathcal{B}[\mathcal{A}()]$.

Note that, using the visual trick for multiplication of matrices of shifting the second matrix upwards and filling the result into the empty gap, we immediately see that something is wrong because we cannot do the dot product. Dimensions do not match. The check of dimensions is built into the algorithm.



However, they COULD be concatenated in the reverse order! $\mathcal{A}[\mathcal{B}()]$.

Indeed: $\begin{pmatrix} 0 & \pi \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & a & 2 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2\pi & \pi \\ a & a^2 + 2 & 2a + 1 \end{pmatrix}$

PROPERTIES OF MATRIX MULTIPLICATION:

Matrix multiplication is not commutative $\mathbf{AB} \neq \mathbf{BA}$ in general

This is fascinating. It tells us that while multiplication of scalars is commutative, when we naturally extend the concept into higher number of dimensions, it ceases to be commutative in general.

This is deeply related to Heisenberg's uncertainty principle in quantum mechanics.

The rest of the properties of addition and multiplication are identical to scalars, but remembering that **multiplication on the left** is in general different to **multiplication on the right**:

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$$
$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$$
$$\lambda(\mathbf{A}+\mathbf{B}) = \lambda\mathbf{A}+\lambda\mathbf{B}$$
$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} = \mathbf{ABC}$$

$\mathbf{A}\lambda\mathbf{B} = \lambda\mathbf{AB}$ (the scalars can always be moved as a common factor)

When not remembering what is and is not allowed with matrices: think of everything you would be allowed to do with scalars. Everything is valid with matrices EXCEPT you always need to keep track whether you multiply on the left, or multiply on the right, and they cannot be interchanged.

The identity matrix $\mathbf{I}$ is the matrix version of the scalar 1. Its diagonal is 1's, rest are 0's. It does not transform vectors. Multiplying $\mathbf{I}$ left or right leaves a matrix unchanged: $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

EXAMPLES OF NON-COMMUTATIVE VS. COMMUTATIVE NESTED LINEAR TRANSFORMATIONS:

20) Let's check that matrix multiplication is non-commutative. Consider the linear transformations $\mathcal{A}$ = projection into the x-axis, and $\mathcal{B}$ = rotation by 90 degrees anticlockwise. Both transformations were associated with a matrix in previous examples as:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

a) Consider the transformations visually. Why are they non-commutative?
b) Consider the transformations mathematically. Calculate the nested transformation matrix $\mathbf{AB}$ and $\mathbf{BA}$. Check they are not equal.

Solution:

a) Visually it is evident that the transformations are not commutative. Projecting into the x axis, and THEN rotating 90 degrees, will mean that all output vectors lie along the y axis.

Rotating by 90 degrees, and THEN projecting into the x-axis, will mean that all output vectors lie along the x-axis.

Both transformations are clearly a different transformation, and so will be represented with a different matrix.

b) $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ corresponds to rotation (B) followed by projection (A).

$\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ corresponds to projection (A) followed by rotation (B).

21) Consider a 90-degree clockwise rotation ($\mathcal{A}$) and a 45-degree anticlockwise rotation ($\mathcal{B}$). Interestingly, the order of the operations is not important in this case. Check that their matrices are commutative and result in a 45-degree clockwise rotation.

Solution:

$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (easily found with the method of previous sections)

$\mathbf{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (matrix multiplication to concatenate the operations)

(the scalars can always be moved as a common factor, as they act on ALL elements of the matrix, they are a global scaling)

$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$

$\mathbf{BA} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$

Same result! The transformation corresponds to the 45 degree rotation clockwise.

This is a rare exception when two matrices give the same result regardless of their order of multiplication. **These matrices are said to commute with each other.**

MATRIX TRANSPOSE AND MATRIX HERMITIAN CONJUGATE

Transpose of a matrix $\mathbf{A}^{\mathrm{T}}$: swap rows by columns

Properties:

$(\lambda\mathbf{A})^{\mathrm{T}} = \lambda\mathbf{A}^{\mathrm{T}}$
$(\mathbf{BA})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}$ (very careful with the order, it must be swapped!)

A square matrix is called **SYMMETRIC** when it is equal to its transpose

22) Find the transpose $\mathbf{A}^{\mathrm{T}}$ of the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}$

Solution:

First row becomes the first column. The second row becomes the second column

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 2 & 1 \end{pmatrix}$$

Think about it as a mirror symmetry across the main centre diagonal

$\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}$

Hermitian conjugate of a matrix:

$\mathbf{A}^\dagger = (\mathbf{A}^*)^\mathrm{T}$ – conjugate each element **and** transpose the matrix.

(the symbol is called 'dagger' †). Note that the **Hermitian inner product** can be written as a matrix product:

$\langle \mathbf{u}|\mathbf{v}\rangle \stackrel{\text{def}}{=} \mathbf{u}^\dagger \mathbf{v}$ if $\mathbf{u}$ and $\mathbf{v}$ written in orthonormal basis

*Note: Above we used the version of Hermitian inner product which has linearity in the second argument, and conjugate linearity in the first, $\langle \mathbf{u}|\mathbf{v}\rangle$. If we wanted to use the version $\langle \mathbf{u}, \mathbf{v}\rangle$ with linearity in the first argument, the expression is uglier: $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^\mathrm{T}\mathbf{v}^*$

23) Find the Hermitian conjugate $\mathbf{A}^\dagger$ of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3i \\ 1+i & i & 0 \end{pmatrix}$

Solution:

Do the transpose, and do the complex conjugate to each element:

$$\mathbf{A}^\dagger = \begin{pmatrix} 1 & 1-i \\ 2 & -i \\ -3i & 0 \end{pmatrix}$$

E. ORTHOGONAL AND UNITARY TRANSFORMATIONS

UNITARY MATRICES

A certain type of transformations have the special property that the length of the vectors is not changed by the transformation and that the angles between vectors are not changed by the transformation: those two statements can be compactly written as $\langle \mathbf{x}|\mathbf{y}\rangle = \langle \mathbf{Ax}|\mathbf{Ay}\rangle$ for any vectors $\mathbf{x}$ and $\mathbf{y}$ (including the possibility of $\mathbf{x} = \mathbf{y}$). A matrix representing such transformation is called a unitary matrix.

$\mathbf{A}$ is a unitary matrix $\Leftrightarrow \langle \mathbf{x}|\mathbf{y}\rangle = \langle \mathbf{Ax}|\mathbf{Ay}\rangle$ for any vectors $\mathbf{x}$ and $\mathbf{y}$

Examples include rotations, reflections, and combinations of both.

We can use the assumption that the transformation does not change lengths to show the following:

$$\langle \mathbf{x}|\mathbf{x}\rangle = \langle \mathbf{Ax}|\mathbf{Ax}\rangle = (\mathbf{Ax})^\dagger(\mathbf{Ax}) = \mathbf{x}^\dagger\mathbf{A}^\dagger\mathbf{Ax}$$

Comparing the leftmost and rightmost terms, we see that $\mathbf{x}^\dagger\mathbf{A}^\dagger\mathbf{Ax} = \mathbf{x}^\dagger\mathbf{x}$, which means that the multiplication $\mathbf{A}^\dagger\mathbf{A} = \mathbf{I}$ "cancels out"**.** This means that the Hermitian conjugate $\mathbf{A}^\dagger$ is equal to the inverse $\mathbf{A}^{-\mathbf{1}}$, a concept introduced formally in the next lesson.

$\mathbf{A}$ is a unitary matrix $\Leftrightarrow \mathbf{A}^\dagger = \mathbf{A}^{-1}$

Also, if we write the matrix multiplication $\mathbf{M} = \mathbf{A}^\dagger\mathbf{A} = \mathbf{I}$ element by element, the matrix elements of $\mathbf{M}$ will correspond to the inner product between every pair of column vectors $\mathbf{a}_i$ in $\mathbf{A}$, that is, $M_{ij} = \langle \mathbf{a}_i|\mathbf{a}_j\rangle$. But we know that the result is the unitary matrix, therefore $M_{ij} = \langle \mathbf{a}_i|\mathbf{a}_j\rangle = \delta_{ij}$ which is the definition of orthonormal set of vectors, so that we can conclude that the columns of $\mathbf{A}$ form an orthonormal set.

$\mathbf{A}$ is a unitary matrix $\Leftrightarrow$ Columns of $\mathbf{A}$ form an orthonormal set

ORTHOGONAL MATRICES:

All the previous properties can be particularised to **matrices which are purely real**. In that case, **the Hermitian conjugate simply becomes the transpose** in all the proofs above, e.g. the inner product becomes the dot product $\langle \mathbf{x}|\mathbf{y}\rangle = \mathbf{x}^\dagger\mathbf{y} = \mathbf{x}^\mathrm{T}\mathbf{y} = \mathbf{x}\cdot\mathbf{y}$, and all results remain identical after that substitution.

The purely real case of unitary matrices is called **orthogonal matrix**.

$\mathbf{A}$ is an orthogonal matrix
$\Leftrightarrow \mathbf{A}^\mathrm{T} = \mathbf{A}^{-1}$
$\Leftrightarrow$ Columns of $\mathbf{A}$ form an orthonormal set of real vectors
$\Leftrightarrow \langle \mathbf{x}|\mathbf{y}\rangle = \langle \mathbf{Ax}|\mathbf{Ay}\rangle$ for any vectors $\mathbf{x}$ and $\mathbf{y}$

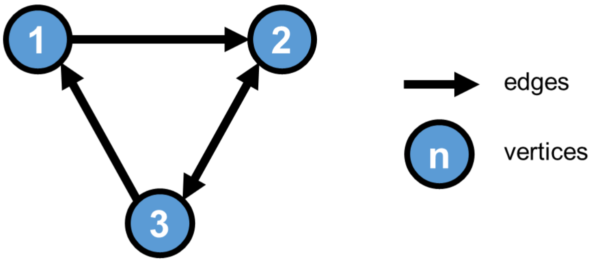
F. POWERS OF A MATRIX (REPEATED MULTIPLICATION)

Calculating powers of a matrix (i.e. multiply a matrix times itself n times) can have interesting applications. This is especially true in fields different to "linear transformation" of vectors.

Application to directed graphs:

For example, given a "directed graph" (a set of vertices connected by edges, in which the edges have directions associated with them), you can construct an associated matrix $\mathbf{A}$ as follows: $a_{ij} = 1$ if there is an edge going from the j-th to the i-th node, otherwise $a_{ij} = 0$. Then, the powers of this matrix $\mathbf{A}^N$ tell you how many ways there is to travel from j to i in exactly N jumps.

24) Obtain the matrix associated to the following connected graph and obtain the total number of ways to travel between nodes in 4 jumps.



$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \qquad \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \mathbf{A}^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{A}^4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

e.g. There are exactly two different ways of getting from 3 to 2 in exactly 4 jumps. Can you find them? We also see it is impossible to travel from 1 to 1 in exactly 4 jumps.

Using a computer, you can go on calculating this in a breeze, e.g. $\mathbf{A}^{20} = \begin{pmatrix} 49 & 65 & 86 \\ 86 & 114 & 151 \\ 65 & 86 & 114 \end{pmatrix}$

There are exactly 49 different ways of getting from 1 to 1 in exactly 20 jumps!

And this was a remarkably simple graph with only 3 nodes. Imagine 100's of nodes: you could model the tube network for London and find number of ways to travel between stations.

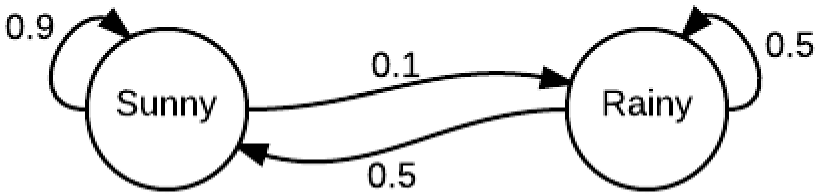
Application to the evolution of a system (Markov chain):

The probabilities of weather conditions (modelled as either rainy or sunny), given the weather on the preceding day, can be represented by a transition matrix:

$$\mathbf{A} = \begin{pmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{pmatrix}$$

Where the element a_{ij} represents the probability that the weather condition is i if the preceding day it was j, with $i, j = 1$ for sunny day and $= 2$ for rainy day. This can be represented as a graph as follows (taken from Wikipedia):

25) Find the following regarding the below Markov process:



a) Given that today is rainy, what is the prediction for 5 days from today?
b) What is the steady-state expected proportion of sunny and rainy days, in the long run?

Solution:

a) The statement that today is rainy can be given as a "state vector" $\mathbf{v}_0 = (0,1)^T$.

The state tomorrow will have probabilities given by: $\mathbf{v_1} = \mathbf{A}\mathbf{v_0} = \begin{pmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

The state on the day after tomorrow: $\mathbf{v_2} = \mathbf{A}\mathbf{v_1} = \mathbf{A^2}\mathbf{v_0} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$

By induction, it is evident that the state for the N-th day will be given by:

$$\mathbf{v_N} = \mathbf{A^N}\mathbf{v_0}$$

Which a computer can calculate extremely fast.

So $\mathbf{v_5} = \mathbf{A^5}\mathbf{v_0} = \begin{pmatrix} 0.83504 & 0.8248 \\ 0.16496 & 0.1752 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8248 \\ 0.1752 \end{pmatrix}$

Notice that, as the power was increased, the two rows of the matrix tended to the same value. That is because the first and second columns represents the expected distribution of weather, after N days, if the initial day was sunny or rainy, respectively. Obviously, as the days go past, the state of the weather today becomes less and less relevant, and the distribution tends to a steady state which depends only on the transition probabilities between the different states. This is usually denoted as $\mathbf{A}^\infty$. Calculating an infinite number of multiplications cannot be done, but there is a smart way to do it.

b) Once the system reaches an equilibrium point, we know that the state will not change after applying one more day, so:

$\mathbf{A}\mathbf{v} = \mathbf{v}$

$\mathbf{Av} = \mathbf{Iv}$

(where $\mathbf{I}$ is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which can be multiplied times a vector or matrix without changing anything)

$\mathbf{Av} - \mathbf{Iv} = \mathbf{0}$

$(\mathbf{A} - \mathbf{I})\mathbf{v} = \mathbf{0}$

$\begin{pmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

This is a system of two equations and two unknowns which can be solved.

Using the usual method to solve a system of equations yields $0 = 0$.

This is because the system of equations is undetermined. Its solution has one degree of freedom. WE can find it by assuming $v_x = \alpha$ and finding $v_y = 0.2\alpha$, solution valid for any value of α. If we now add the condition that $v_x + v_y = 1$, since they are probabilities, we get that the steady state is $(v_x, v_y) = (0.833 \dots, 0.166 \dots)$

OTHER APPLICATIONS

26) **Example in Physics**: General Lorentz Transformation (special relativity) seen as a matrix linear transformation. Taken from Prof. Victor Yakovenko notes *"In most textbooks, the Lorentz transformation is derived from the two postulates: the equivalence of all inertial reference frames and the invariance of the speed of light. However, the most general transformation of space and time coordinates can be derived using only the equivalence of all inertial reference frames and the symmetries of space and time."*

http://www2.physics.umd.edu/~yakovenk/teaching/Lorentz.pdf

Derivation of Lorentz transformation from first principles. Very natural/fundamental assumptions:

0) From translational symmetry of space and time, transformation between coordinates systems must be a linear transformation:

From translational symmetry of space and time, the **relative** distances between two events in one reference frame must depend only on the **relative** distances in another frame: $x_2' - x_1' = f_x(x_2 - x_1, t_2 - t_1)$ and $t_1' - t_2' = f_t(x_2 - x_1, t_2 - t_1)$. Because these equations must be valid for any two events, the functions f_x and f_t must be linear functions. **Therefore**: Consider reference system origin O with (x, t) and reference system origin O' at relative speed v with respect to O, coordinates (x', t'). The relation must be linear, hence can be described by a matrix:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ t \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

1) Definition of relative speed between the two systems:

1a) $x' = 0 \rightarrow x = vt$ **Therefore**: $\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} vt \\ t \end{pmatrix} \rightarrow b = -av$

1b) $x = 0 \rightarrow x' = -vt'$ **Therefore**: $\rightarrow \begin{pmatrix} -vt' \\ t' \end{pmatrix} = \begin{pmatrix} a & -av \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} \rightarrow t' = dt \rightarrow -vdt = -vat \rightarrow a = d$

Therefore, our matrix must always have certain relations between its elements:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} a & -av \\ c & a \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} = a\begin{pmatrix} 1 & -v \\ c/a & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} = \gamma_v \begin{pmatrix} 1 & -v \\ F_v & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

2) Combination of two transformations must also be itself a transformation:

$$\mathbf{A}_{v2}\mathbf{A}_{v1} = \mathbf{A}_v$$

$$\mathbf{A}_{v2}\mathbf{A}_{v1} = \gamma_{v1}\gamma_{v2}\begin{pmatrix} 1 - F_{v1}v_2 & -v_1 - v_2 \\ F_{v1} + F_{v2} & 1 - F_{v1}v_2 \end{pmatrix} \text{ must also be of the form } \gamma_v \begin{pmatrix} 1 & -v \\ F_v & 1 \end{pmatrix}$$

Therefore: $F_{v1}v_2 = F_{v2}v_1 \to \frac{F_{v1}}{v_1} = \frac{F_{v2}}{v_2}$ for any v_i $\quad \to F_{vi} = (const)v_i = \frac{v_i}{\alpha^2}$

3) Transformations with opposite velocities must bring us back to the original system:

$$\mathbf{A}_v\mathbf{A}_{-v} = \mathbf{I}$$

$$\mathbf{A}_v\mathbf{A}_{-v} = \gamma_v\gamma_{-v}\begin{pmatrix} 1 & -v \\ v/\alpha^2 & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ -v/\alpha^2 & 1 \end{pmatrix} = \gamma_v\gamma_{-v}\begin{pmatrix} 1 + \frac{v^2}{\alpha^2} & 0 \\ 0 & 1 + \frac{v^2}{\alpha^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\to \quad \gamma_v\gamma_{-v} = \frac{1}{1 + v^2/\alpha^2}$$

Because of symmetry of space, the function γ_v must depend only on the absolute value of velocity:

$$\gamma_v = \frac{1}{\sqrt{1 + \frac{v^2}{\alpha^2}}}$$

From those 4 assumptions, we have arrived at the general form of the transformation:

$$\boxed{\begin{pmatrix} x' \\ t' \end{pmatrix} = \frac{1}{\sqrt{1 + v^2/\alpha^2}}\begin{pmatrix} 1 & -v \\ v/\alpha^2 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}}$$

The ONLY free parameter to determine is α, which has dimensions of speed.

Galileo and Newton would have said "addition of velocities" (who can blame them) as a strong version of (2): $\mathbf{A}_{v1}\mathbf{A}_{v2} = \mathbf{A}_{v1+v2}$. This results in $1/\alpha^2 = 0$ giving the Galilean transformations.

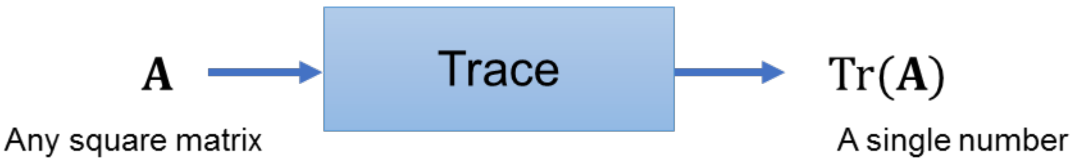
Einstein instead would only agree on $\mathbf{A}_v\mathbf{A}_{-v} = \mathbf{A}_0$ as strictly necessary (which is (3)) and then used the extra degree of freedom allowed by this to say "a light ray is seen the same in any frame" to obtain a value for α:

$$x = ct \to x' = ct' \quad \Rightarrow \quad \begin{pmatrix} ct' \\ t' \end{pmatrix} = \mathbf{A}_v\begin{pmatrix} ct \\ t \end{pmatrix} \to \alpha = \pm ic \to \alpha^2 = -c^2$$

2.2 MATRIX TRACE AND DETERMINANT

There are two operations that act on matrices and result in a useful scalar related to some important properties of that matrix: the trace, and the determinant. They can only act on **square matrices**.

A. TRACE OF A MATRIX



The trace $\mathrm{Tr}(\mathbf{A})$ is equal to the sum of the elements in the diagonal.

Properties:

Trace is a linear operation: $\mathrm{Tr}(\lambda\mathbf{A}+\mu\mathbf{B}) = \lambda\mathrm{Tr}(\mathbf{A})+\mu\mathrm{Tr}(\mathbf{B})$
Trace of the product is independent of the order: $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$
Trace of the identity matrix equals its dimension: $\mathrm{Tr}(\mathbf{I}) = \dim(\mathbf{I})$

It can be shown that the ONLY operator which fulfils the three properties above is the sum of the elements of the diagonal. In fact, we could say that the trace is defined by these properties.

1) Find the trace of matrices $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $\mathbf{AB}$, and $\mathbf{BA}$

$\mathrm{Tr}(\mathbf{A}) = \mathrm{Tr}\left[\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\right] = 3;$ $\mathrm{Tr}(\mathbf{B}) = \mathrm{Tr}\left[\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\right] = 0;$

$\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}\left[\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}\right] = 1$

$\mathrm{Tr}(\mathbf{BA}) = \mathrm{Tr}\left[\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}\right] = 1$

As expected, $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$.

B. MATRIX DETERMINANT

Beautiful explanation in 3blue1brown YouTube channel:
[The determinant | Essence of linear algebra, chapter 6] (10 min)

The determinant of a matrix tells us about the associated linear transformation:

For a 1D linear transformation ($y = ax$) it tells us how lengths are scaled, i.e. $\det(a) = a$,
For a 2D linear transformation ($\mathbf{b} = \mathbf{Av}$) it tells us how areas are scaled (area of parallelogram formed by transformed basis),
For a 3D linear transformation ($\mathbf{b} = \mathbf{Av}$) it tells us how volumes are scaled (volume of parallelepiped formed by transformed basis). etc.

The determinant is negative if the length/area/volume/… is "flipped".

The determinant is zero if the transformation "squashes" the input vector space into an output space of smaller dimensions (e.g. projection of 2D space into a line, projection of 3D space into a plane or line, etc.)

C. CALCULATION OF THE DETERMINANT: GENERAL AND SIMPLE CASES

For 2×2 matrices: $\det\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = ad - bc$

It is equal to the oriented **area of the parallelogram** enclosed by the two column vectors.

For 3×3 matrices: $\det\left[\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right] = a\left|\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right| - b\left|\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right| + c\left|\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right|$

It is equal to the oriented **volume of the parallelepiped** enclosed by the three column vectors of the matrix.

For $N \times N$ matrices (general recipe):

Consider a square matrix $\mathbf{A}$ with elements a_{ij}. The determinant is equal to the oriented N-dimensional "volume" of the columns. It is defined in terms of determinants of $(N-1) \times (N-1)$ matrices.

1. Calculate the **minors** of the matrix M_{ij} for **any** row or column.
2. Calculate the **cofactors** of the matrix C_{ij} for that **same** row or column.
3. Sum the products of the elements of that row or column with each corresponding cofactor: e.g. $a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$ (fixed $i = 2$)
or $a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$ (fixed $j = 3$).

Definition: The **minor** M_{ij} of the element a_{ij} of an $N \times N$ matrix $\mathbf{A}$ is equal to the determinant of the $(N-1) \times (N-1)$ matrix that results when we remove the *i*-th row and *j*-th column of $\mathbf{A}$.

Example: Calculate all the minors M_{ij} of the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

To calculate M_{ij} cross out the row and column corresponding to the element a_{ij}. That is, the i-th row, and the j-th column. The minor is the determinant of the matrix that is left.

$$M_{11} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}\right] = a_{22}a_{33} - a_{23}a_{32}$$

$$M_{12} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}\right] = a_{21}a_{33} - a_{23}a_{31}$$

$$M_{13} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\right] = a_{21}a_{32} - a_{22}a_{31}$$

$$M_{21} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}\right] = a_{12}a_{33} - a_{13}a_{32}$$

$$M_{22} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}\right] = a_{11}a_{33} - a_{13}a_{31}$$

$$M_{23} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}\right] = a_{11}a_{32} - a_{12}a_{31}$$

$$M_{31} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}\right] = a_{12}a_{23} - a_{13}a_{22}$$

$$M_{32} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}\right] = a_{11}a_{23} - a_{13}a_{21}$$

$$M_{33} = \det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = a_{11}a_{22} - a_{12}a_{21}$$

Definition: $C_{ij} = (-1)^{i+j} M_{ij}$. The **cofactor** C_{ij} associated with the minor M_{ij} is equal to the minor M_{ij} with a flip in sign given by $(-1)^{i+j}$, which is a chessboard-like array of +1 and -1.

Example: This is what the term $(-1)^{i+j}$ looks like for all elements of a 3×3 matrix:

$$(-1)^{i+j} = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Therefore, a matrix containing all cofactors, usually called $\mathbf{C}$, looks like this:

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix}$$

Calculation of determinants becomes messy and slow for matrices greater than 3×3. We should always choose the row or column wisely to make the calculation easier (normally the one which has a greater number of zeroes).

2) Example: Calculate the determinant of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 2 & 2 & 1 & 0 \\ 3 & 0 & -2 & 1 \end{pmatrix}$

Let's choose the second column, because all elements except one are zero.

The determinant will be the sum of the products of the elements of that column with each corresponding cofactor:

$$\det(\mathbf{A}) = 0\,C_{12} + 0\,C_{22} + 2\,C_{32} + 0\,C_{42}$$

$$= -0\,M_{12} + 0\,M_{22} - 2\,M_{32} + 0\,M_{42}$$

$$= -2\,M_{32} = -2\det\left[\begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 3 & -2 & 1 \end{pmatrix}\right]$$

Which in turn can be calculated by choosing the second row (remember the sign $(-1)^{i+j}$)

$$= -2\left(-0 + 2\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}\right)$$

$$= -2(2(1+3) - 1|(-2-6)|)$$

$$= -2(8+8) = -32$$

SIMPLE CASES

The determinant of **diagonal** or **triangular** matrix is equal to the **product of the elements of the diagonal**.

Diagonal matrix: A diagonal matrix is one whose off-diagonal elements are all zero. It corresponds to scaling of axes.

Example: $\det\left[\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \lambda+2 \end{pmatrix}\right] = 2(-4)(\pi)(\lambda+2) = -8\pi\lambda - 16\pi$

Triangular matrix (upper/lower triangular): An upper/lower triangular matrix has zeroes in all elements below/above the diagonal. It corresponds to scaling and shear of axes. The area remains the same as if it was only scaling.

Example: Upper triangular matrix: $\det\left[\begin{pmatrix} 1 & 3 & 2 \\ 0 & -4 & 1 \\ 0 & 0 & 2 \end{pmatrix}\right] = 1(-4)(2) = -8$

Example: Lower triangular matrix: $\det\left[\begin{pmatrix} 1 & 0 & 0 \\ 7 & -4 & 0 \\ 3 & 1 & 2 \end{pmatrix}\right] = 1(-4)(2) = -8$

D. PROPERTIES OF DETERMINANTS

Properties related to matrix operations:

- **Determinant is NOT linear** $\det(\lambda\mathbf{A}+\mu\mathbf{B}) \neq \lambda\det(\mathbf{A})+\mu\det(\mathbf{B})$
- **Determinant of a product**: $\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det(\mathbf{A})\det(\mathbf{B})$
  *Visually: the combination of two transformations always changes the area in the **same** way regardless of the order of the transformations.*
- **Determinant of the transpose**: $\det(\mathbf{A}^{\mathrm{T}}) = \det(\mathbf{A})$.
- **Determinant of the complex conjugate**: $\det(\mathbf{A}^*) = (\det(\mathbf{A}))^*$

Properties related to changes in the vectors forming the columns and rows of a matrix:

It is easy to interpret these properties in terms of the vectors forming the columns of the matrix, which are the transformed version of the input basis $\mathcal{A}(\mathbf{e_i})$. But the same properties apply to the row vectors, too, as follows from the transpose property.

- **If the vectors of a matrix are linearly dependent, the determinant of the matrix is zero**
  - *The transformation squashes N-dimensional space into a space of lower dimension (equal to the dimension of the span of the vectors)*
  - If two vectors are proportional to each other, the determinant is zero *(duh)*
- **Interchange of vectors**: Interchanging any two vectors in the matrix flips the sign of the determinant (*as the transformed grid is flipped*) but leaves the magnitude unchanged.
- **Common factors:** You can remove a common factor λ to any vector in the matrix, and the remaining determinant just needs to be multiplied by λ.
  e.g. $$\det\left[\begin{pmatrix} 2 & 5 & 0 \\ 3 & -20 & 1 \\ 1 & 15 & 2 \end{pmatrix}\right] = 5\det\left[\begin{pmatrix} 2 & 1 & 0 \\ 1 & -4 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right]$$
  - *Visually: This scales the transformed grid along one of its directions by a factor λ.*
  - By extension, doing it to all columns: $|\lambda\mathbf{A}| = \lambda^N|\mathbf{A}|$.
- **Linear combination**: You can add, to any vector, a linear combination of the other vectors, and the determinant is unchanged! (very powerful property!)
  - $\mathbf{v}_i + \lambda\mathbf{v}_j \to \mathbf{v}_i$ (for any λ and any $j \neq i$) does not change the determinant
  - *Visually: This is like introducing a shear into a given transformation. The areas/volumes are unaffected by shears (area of parallelepiped unchanged)*

The determinant can be elegantly **defined** via a selection of its properties.

The determinant of a matrix $\mathbf{A}$ is the **unique** function that satisfies:
1) $\det(\mathbf{A}) = 0$ when two columns are equal.
2) The determinant is linear in the columns.
3) if $\mathbf{I}$ is the identity, $\det(\mathbf{I}) = 1$.

You can easily convince yourself that the oriented volume $\mathrm{vol}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ between vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a function that satisfies exactly those same properties if we place the vectors as the columns of a matrix $\mathbf{A} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$. Hence $\mathrm{vol}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det(\mathbf{A})$.

Using the properties of determinants can be very useful to calculate determinants because they allow you to modify the matrix and turn it into a triangular matrix, for example, in which calculation of the determinant is trivial.

3) **Example**: Calculate $\det\left[\begin{pmatrix} 2 & 2 & 0 \\ 1 & -3 & 4 \\ 1 & 3 & 2 \end{pmatrix}\right]$ by using properties of determinants to turn the matrix into a triangular one.

Modify the second row by adding to it (-1/2) of the first one. We write that operation as $R_2 - \left(\frac{1}{2}\right) R_1 \rightarrow R_2$. Continue doing similar operations which leave the determinant unchanged:

$$\det\left[\begin{pmatrix} 2 & 2 & 0 \\ 1 & -3 & 4 \\ 1 & 3 & 2 \end{pmatrix}\right] \xrightarrow{R_2-\left(\frac{1}{2}\right)R_1\rightarrow R_2} = \det\left[\begin{pmatrix} 2 & 2 & 0 \\ 0 & -4 & 4 \\ 1 & 3 & 2 \end{pmatrix}\right] \xrightarrow{R_3-\left(\frac{1}{2}\right)R_1\rightarrow R_3}$$

$$= \det\left[\begin{pmatrix} 2 & 2 & 0 \\ 0 & -4 & 4 \\ 0 & 2 & 2 \end{pmatrix}\right] \xrightarrow{R_3+\left(\frac{1}{2}\right)R_2\rightarrow R_3} = \det\left[\begin{pmatrix} 2 & 2 & 0 \\ 0 & -4 & 3 \\ 0 & 0 & 4 \end{pmatrix}\right]$$

$$= (\text{this is a triangular matrix}) = (2)(-4)(4) = -32$$

Problems:

4) Find the determinant $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 3 \\ 1 & 1 & 1 \end{vmatrix}$

Solution: Two columns are equal, therefore the determinant is zero.

5) Solve the following equation with respect to x:

$$\begin{vmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{vmatrix} = 0$$

Solution: Let's expand the determinant along the 1st row:

$$\begin{vmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{vmatrix} = x\begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & x & 1 \\ 1 & 1 & x \end{vmatrix} + 0 - 1\begin{vmatrix} 1 & x & 1 \\ 0 & 1 & x \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$x[x(x^2-1) - 1(x-0) + 0] - 1[1(x^2-1) - 1(0-1) + 0] - 1[1(1-0) - x(0-x) + 1(-1)]$$

$$\rightarrow x(x^3 - 2x) - 1(x^2 - 1 + 1) - 1(1 + x^2 - 1) = 0$$

$$\rightarrow x^4 - 2x^2 - x^2 - x^2 = 0$$

$$\rightarrow x^4 - 4x^2 = x^2(x^2 - 4) = 0$$

$$\rightarrow \{x_1 = 0, x_{2,3} = \pm 2\}$$

2.3 MATRIX INVERSE, RANGE AND NULL SPACE

Beautiful introduction in 3blue1brown YouTube channel:
Inverse matrices, column space and null space | Essence of linear algebra ch. 7 (12 min)

A. SOLVING THE INVERSE TRANSFORMATION

So far, I have given you $\mathbf{x}$ and asked you to find $\mathbf{v} = \mathbf{A}\mathbf{x}$.

What if I give you the output $\mathbf{v}$ and ask you to solve for $\mathbf{x}$? How do we solve $\mathbf{A}\mathbf{x} = \mathbf{v}$ for $\mathbf{x}$? This is the important **inverse problem**.

Consider the simplest case: 1D-to-1D transformations. A linear transformation is a multiplication

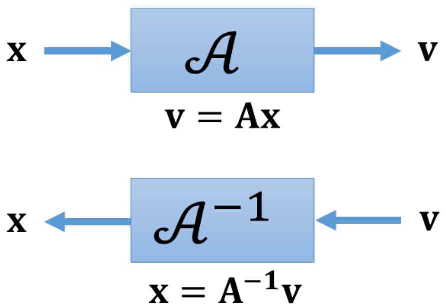
$v = ax$ (where a is a scalar playing the role of the 1×1 matrix)

We can find the value of x given a value for v by dividing both sides by a:

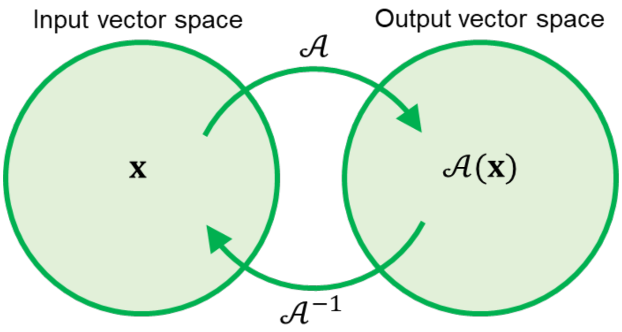
$x = a^{-1}v$ (where $a^{-1} = (1/a)$ is a scalar playing the role of a different 1×1 matrix called inverse)

We can see this as an inverse transformation. A transformation which undoes the previous one.

In higher dimensions, the inverse transformation is represented by a matrix which we call the inverse matrix, denoted as $\mathbf{A}^{-1}$.



Evidently, if we apply the transformation $\mathbf{A}$ and follow it by the transformation $\mathbf{A}^{-1}$, we will get back to the original vector.



$$\mathbf{x} = \mathcal{A}^{-1}[\mathcal{A}(\mathbf{x})] \;\rightarrow\; \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

This means that $\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{x}$, or in other words, $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix which represents the unitary transformation that leaves everything unchanged.

TRIVIAL EXAMPLES OF INVERSE TRANSFORMATION

Scaling: Consider a transformation which scales the x-direction by 3 and the y-direction by ½. The inverse transformation should undo the previous one, therefore it should scale the x-direction by $1/3$ and the y-direction by 2.

Indeed, the matrices fulfil the condition:

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix};\ \mathbf{A^{-1}} = \begin{pmatrix} 1/3 & 0 \\ 0 & 2 \end{pmatrix};\ \mathbf{A^{-1}A} = \begin{pmatrix} 1/3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rotations: Consider a 2D rotation by θ degrees clockwise. The inverse transformation should be an identical rotation but anticlockwise, i.e. with a reverse sign in θ.

$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$; rotation θ clockwise

$\mathbf{A^{-1}} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$; rotation θ anti-clockwise

Indeed: $\mathbf{A^{-1}A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$
$\begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

CALCULATION OF THE INVERSE – GENERAL RECIPE (IF IT EXISTS)

Given a matrix $\mathbf{A}$ with elements a_{ij}.

We calculate the matrix $\mathbf{C}$ containing the cofactors C_{ij} as defined earlier.

The transpose of the matrix of cofactors $\mathbf{C}^T$ will have elements $D_{ij} = C_{ji}$. (Note the swap $i \leftrightarrow j$)

The inverse matrix $\mathbf{A^{-1}}$ will have elements b_{ij} which can be calculated as:

$$b_{ij} = \frac{C_{ji}}{\det(\mathbf{A})}$$

Or, writing the matrix explicitly:

$$\mathbf{A^{-1}} = \frac{\mathbf{C}^T}{\det(\mathbf{A})}$$

For a 2×2 matrix, the recipe can be written very simply:

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

This matrix fulfils $\mathbf{A^{-1}A} = \mathbf{AA^{-1}} = \mathbf{I}$

1) Find the inverse matrix for $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

Solution:

First find the determinant. Since it is a triangular matrix, the determinant is the product of the diagonal, so $\det(\mathbf{A}) = 2$.

Now find the minors:

$$M_{11} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2; \quad M_{12} = \begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} = 0; \quad M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{21} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4; \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2; \quad M_{23} = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{31} = \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = -2; \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1; \quad M_{23} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

Hence the matrix of minors is:

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ -2 & -1 & 1 \end{pmatrix}$$

The matrix of cofactors is equal to the matrix of minors but changing sign of the terms with odd $(i + j)$ i.e. a chessboard-like pattern of signs:

$$\mathbf{C} = \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ -M_{31} & -M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} 2 & -0 & 0 \\ -4 & 2 & -0 \\ -2 & +1 & 1 \end{pmatrix}$$

Finally, we just need to transpose this matrix, and divide by the determinant:

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det(\mathbf{A})} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 2 & -4 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$$

We can always check if this is the inverse by doing the multiplication:

$$\mathbf{A^{-1}A} = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{AA^{-1}} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) Find the inverse matrix for $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Solution:

First find the determinant: this one is a particularly easy matrix. Add the 3 positive and 3 negative diagonals:

$$\det(\mathbf{A}) = 1 + 0 + 0 - 0 - 1 - 1 = -1.$$

Now find the minors:

$$M_{11} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0; \quad M_{12} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1; \quad M_{13} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$M_{21} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1; \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \quad M_{23} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$M_{31} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1; \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1; \quad M_{23} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Hence the matrix of minors is:

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The matrix of cofactors is equal to the matrix of minors but changing sign of the terms with odd $(i + j)$:

$$\mathbf{C} = \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ -M_{31} & -M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Finally, we just need to transpose this matrix, and divide by the determinant:

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det(\mathbf{A})} = \frac{1}{-1} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Finally, we can always check if this is the inverse by doing the multiplication:

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The algorithm becomes very tedious for $N = 4$ and becomes impractical for even higher dimensions.

In real life, no-one uses the inverse to solve a system $\mathbf{Ax} = \mathbf{b} \rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. We will study a practical method in the next chapter.

INVERSE OF UNITARY (ORTHOGONAL) MATRICES

In chapter 2.1 we saw that unitary matrices (those matrices whose linear transformation does not change lengths nor angles) have a very easy to calculate inverse:

$\mathbf{A}$ is a unitary matrix $\Leftrightarrow$ Columns of $\mathbf{A}$ form an orthonormal set $\Leftrightarrow \mathbf{A}^{-1} = \mathbf{A}^{\dagger}$

As we also saw, if a unitary matrix is purely real, then it is called an orthogonal matrix, and its inverse is simply its transpose!

$\mathbf{A}$ is an orthogonal matrix $\Leftrightarrow$ Columns of $\mathbf{A}$ form a purely real orthonormal set $\Leftrightarrow \mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$

3) **Example**: Calculate the inverse of the matrix $\mathbf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$

Solution: Notice that the columns of $\mathbf{A}$ form an orthonormal set: $\left\langle \frac{1}{\sqrt{2}}(1, i, 0), \frac{1}{\sqrt{2}}(1, i, 0) \right\rangle =$
$\left\langle \frac{1}{\sqrt{2}}(1, -i, 0), \frac{1}{\sqrt{2}}(1, -i, 0) \right\rangle = \langle (0,0,1), (0,0,1) \rangle = 1$

$\left\langle \frac{1}{\sqrt{2}}(1, i, 0), \frac{1}{\sqrt{2}}(1, -i, 0) \right\rangle = \left\langle \frac{1}{\sqrt{2}}(1, i, 0), (0,0,1) \right\rangle = \left\langle \frac{1}{\sqrt{2}}(1, -i, 0), (0,0,1) \right\rangle = 0.$

So we are lucky! This is a unitary matrix, and therefore its inverse is simply its Hermitian conjugate:

$$\mathbf{A}^{-1} = \mathbf{A}^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

4) Calculate the inverse transformation of $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Solution:

We can see that the two columns of the matrix are orthonormal vectors:

$$\left\langle \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\rangle = 0;$$

$$\left\langle \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\rangle = \sin^2\theta + \cos^2\theta = 1$$

Therefore, **$\mathbf{A}$ is an orthogonal matrix,** and so by definition its inverse is simply its transpose:

$\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$

Indeed: $\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$
$\begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

B. CASES WITH NO INVERSE: SINGULAR MATRICES

Trivial example: the transformation $\mathbf{A} = \mathbf{0}$ has no inverse

Consider the 1D transformation given by $v = ax$ with $a = 0$. This transformation collapses the entire number line into the origin. Given an output $v = 0$, we have no idea what the input was. This transformation has no inverse. Indeed, we cannot divide both sides by a because $a = 0$.

The solution to $v = 0x = 0$ is that $x = \lambda$, for any value of λ.

The solution to $v = 0x = 2$ is that there are no solutions. NO possible input x gives $v = 0x = 2$.

This 1D scenario extends nicely to N-dimensions, when $\mathbf{v} = \mathbf{A}\mathbf{x}$ and $\mathbf{A} = \mathbf{0}$.

In higher number of dimensions, there are more interesting possibilities for transformations with no inverse.

When $\det(\mathbf{A}) = 0$ the matrix has no inverse. There exists no matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

A matrix $\mathbf{A}$ with $\det(\mathbf{A}) = 0$ is called a <u>**singular matrix**</u>.

Such transformations always map the input N-dimensional space into a subspace of dimension lower than N, therefore, they always map an entire subspace of the input into the origin.

Example - Projection of 2D space into the x-axis:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \det \mathbf{A} = 0$$

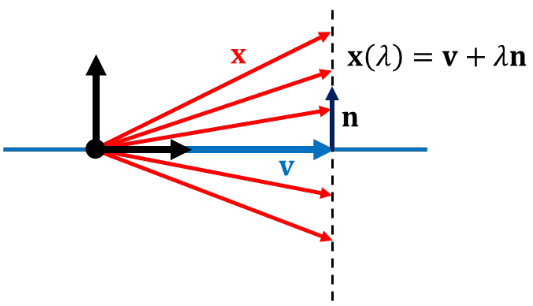
Solve $\mathbf{v} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $\mathbf{x}$.

Let's solve the equation $\mathbf{v} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ for $\mathbf{x}$.

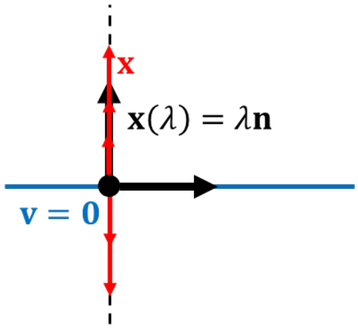
There is no inverse transformation, because there are infinite possible input values.

In this example, the infinite valid solutions to the problem $\mathbf{v} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ form a line:

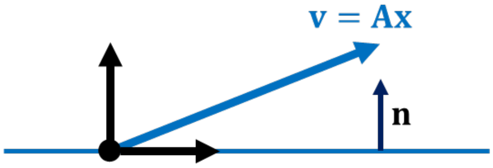
$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is a line, the solution has 1 degree of freedom}$$



Notice something: the second term of the solution $\lambda\mathbf{n}$ is exactly the solution of the "homogeneous system" $\mathbf{A}\mathbf{x} = 0$ (i.e. when the output $\mathbf{v} = \mathbf{0}$ is in the origin).



Now let's try to solve $\mathbf{v} = \mathbf{Ax} = (2,1)^T$. This has no solutions because $\mathbf{v}$ is not inside the space of allowed outputs of the transformation.

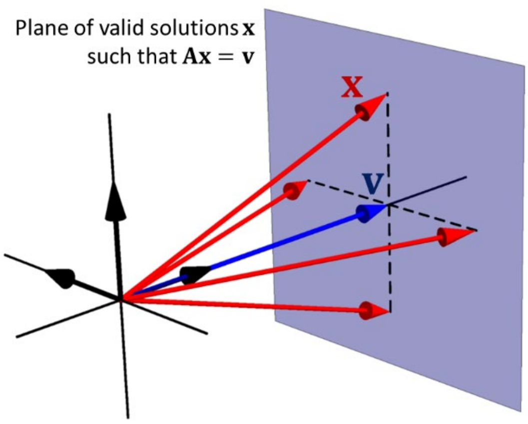


Example - projection of 3D space into the x-axis:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \det \mathbf{A} = 0$$

Solve $\mathbf{v} = \mathbf{Ax} = (3,0,0)^T$ for $\mathbf{x}$.

Let's solve the problem $\mathbf{v} = \mathbf{Ax} = (3,0,0)^T$. What are the input vectors $\mathbf{x}$ which get projected onto $\mathbf{v}$?



$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \underbrace{\lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{infinite plane}}$$ The solution has 2 degrees of freedom.

The range of possible outputs is a line, it has 1 degree of freedom.

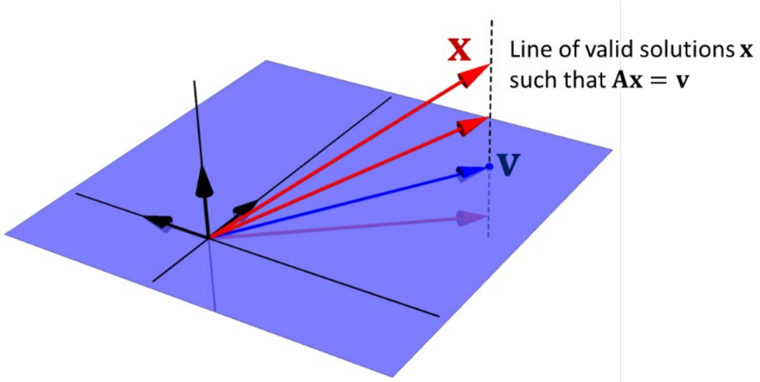
Once again, notice that the term $\lambda\hat{\mathbf{y}} + \mu\hat{\mathbf{z}}$ is exactly the solution to the homogeneous equation $\mathbf{v} = \mathbf{Ax} = \mathbf{0}$. Not a coincidence!

Example: Projection of 3D space into $z = 0$ plane:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \det \mathbf{A} = 0$$

Solve $\mathbf{v} = \mathbf{Ax} = (3, -2, 0)^T$ for $\mathbf{x}$.

Let's solve the problem $\mathbf{v} = \mathbf{Ax} = (3, -2, 0)^T$.



$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \underbrace{\lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{infinite line}}$$

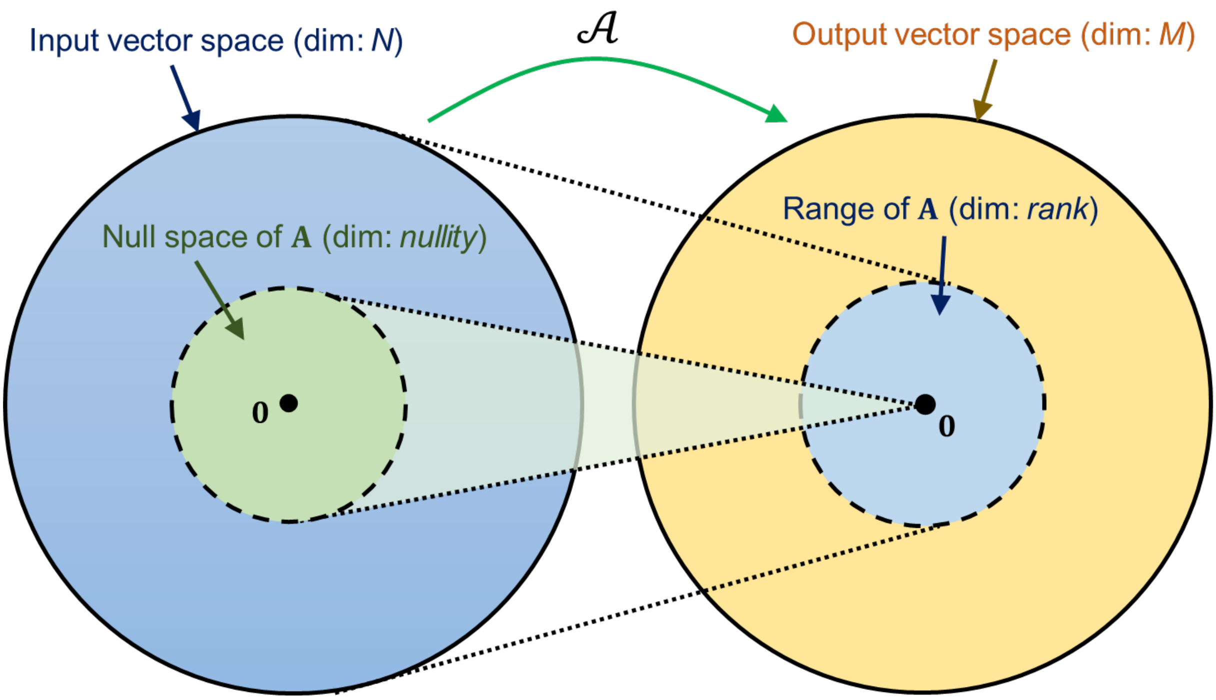
The solution has 1 degree of freedom.

The range of possible outputs is a plane, it has 2 degrees of freedom.

C. RANGE AND NULL SPACE OF A TRANSFORMATION

There is a beautiful pattern hidden in the previous examples!

This figure and table (explained in the next page) summarizes the general case:



Theorem of dimensions: $\text{rank}(\mathbf{A}) = N - \text{nullity}(\mathbf{A})$

Name of space/subspace	Dimension	Span
Input space	N	$\text{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_N\}$ basis vectors of input space
Range of $\mathbf{A}$ or Column Space of $\mathbf{A}$	$\text{rank}(\mathbf{A})$	$\text{span}\{\mathcal{A}(\mathbf{e}_1), \cdots, \mathcal{A}(\mathbf{e}_N)\}$ columns of matrix
Null Space of $\mathbf{A}$	$\text{nullity}(\mathbf{A})$	$\text{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_{\text{Nullity}}\}$ vector solutions to $\mathbf{Ax} = \mathbf{0}$

RANGE (OR COLUMN SPACE) OF $\mathbf{A}$:

The transformation $\mathbf{A}$ will map the input vector space into a subspace of the output vector space, which may be the entire space. This subspace is called the range of $\mathbf{A}$. It is also called the column space of $\mathbf{A}$, because it is the subspace spanned by the columns of $\mathbf{A}$. This is because the transformed vectors $\mathbf{Ax}$ for every possible $\mathbf{x}$ can be written as:

$$\mathbf{Ax} = \begin{pmatrix} | & | & & | \\ \mathcal{A}(\mathbf{e}_1) & \mathcal{A}(\mathbf{e}_2) & \dots & \mathcal{A}(\mathbf{e}_N) \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1\mathcal{A}(\mathbf{e}_1) + x_2\mathcal{A}(\mathbf{e}_2) + \cdots + x_N\mathcal{A}(\mathbf{e}_N)$$

$$= \mathrm{span}\{\mathcal{A}(\mathbf{e}_1), \mathcal{A}(\mathbf{e}_2), \cdots, \mathcal{A}(\mathbf{e}_N)\} \text{ for any possible } \mathbf{x}.$$

Range of $\mathbf{A}$ = $\mathrm{span}\{\mathcal{A}(\mathbf{e}_1), \mathcal{A}(\mathbf{e}_2), \cdots, \mathcal{A}(\mathbf{e}_N)\}$ where $\mathcal{A}(\mathbf{e}_i)$ are the columns of $\mathbf{A}$, of which some may be redundant in terms of their span.

The dimension of the range of $\mathbf{A}$ is called the **rank** of $\mathbf{A}$.

If the columns are linearly independent, $\mathrm{rank}(\mathbf{A}) = N$, and the matrix is said to have *full rank*. This happens only when $\det(\mathbf{A}) \neq 0$.

NULL SPACE OF $\mathbf{A}$:

If the transformation $\mathbf{A}$ is singular, there will exist a subspace of the input vector space which is mapped to $\mathbf{0}$. This is called the null space of $\mathbf{A}$.

The null space is the set of solutions for $\mathbf{x}$ to the equation $\mathbf{Ax} = 0$.

Null space = $\mathrm{span}\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_{\mathrm{Nullity}}\}$ where $\{\mathbf{z}_i\}$ are linearly independent vectors that map to zero: $\mathbf{Az}_i = \mathbf{0}$

The dimension of the null space of $\mathbf{A}$ is called the **nullity** of $\mathbf{A}$.

THEOREM OF DIMENSIONS:

Theorem of dimensions: $\mathrm{rank}(\mathbf{A}) = N - \mathrm{nullity}(\mathbf{A})$

Name of space/subspace	Dimension	Span
Input space	N	$\mathrm{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_N\}$ basis vectors of input space
Range of $\mathbf{A}$ or Column Space of $\mathbf{A}$	$\mathrm{rank}(\mathbf{A})$	$\mathrm{span}\{\mathcal{A}(\mathbf{e}_1), \cdots, \mathcal{A}(\mathbf{e}_N)\}$ columns of matrix
Null Space of $\mathbf{A}$	$\mathrm{nullity}(\mathbf{A})$	$\mathrm{span}\{\mathbf{z}_1, \cdots, \mathbf{z}_{\mathrm{Nullity}}\}$ vector solutions to $\mathbf{Ax} = \mathbf{0}$

The null space $\mathbf{Ax} = \mathbf{0}$ is very useful when solving the inverse problem $\mathbf{Ax} = \mathbf{v}$:

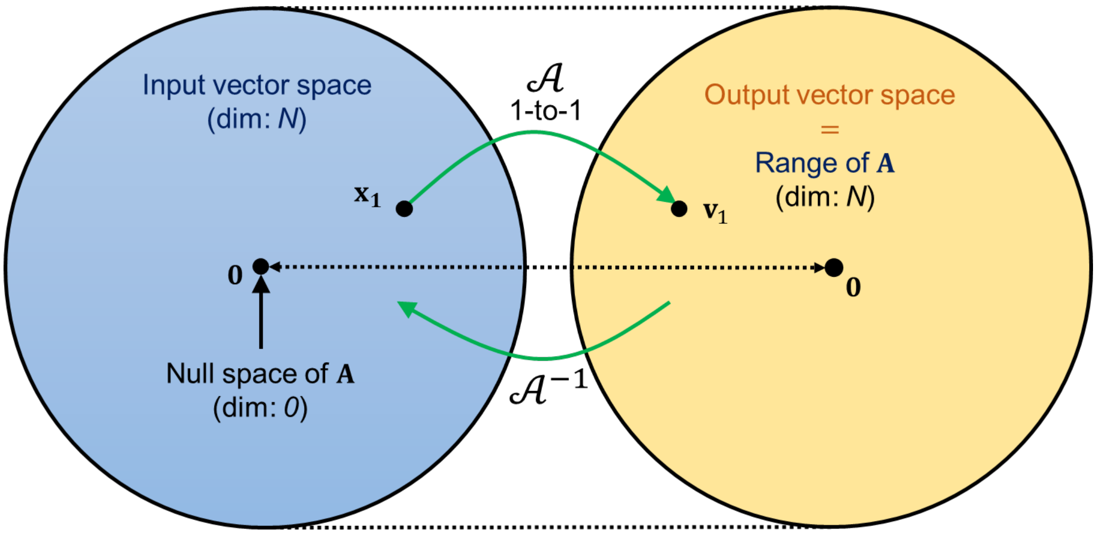
If we find a solution $\mathbf{x_1}$ such that $\mathcal{A}(\mathbf{x}_1) = \mathbf{v}$
then the set of vectors $\mathbf{x}_1 + \text{Null space}(\mathbf{A})$ are all a solution too, due to linearity.

Proof: $\mathcal{A}\big(\mathbf{x}_1 + \text{Null space}(\mathbf{A})\big) = \underbrace{\mathcal{A}(\mathbf{x}_1)}_{\mathbf{v}} + \underbrace{\mathcal{A}\big(\text{Null space}(\mathbf{A})\big)}_{\mathbf{0}} = \mathbf{v}$

Non-singular square matrix ($\det(\mathbf{A}) \neq 0$):

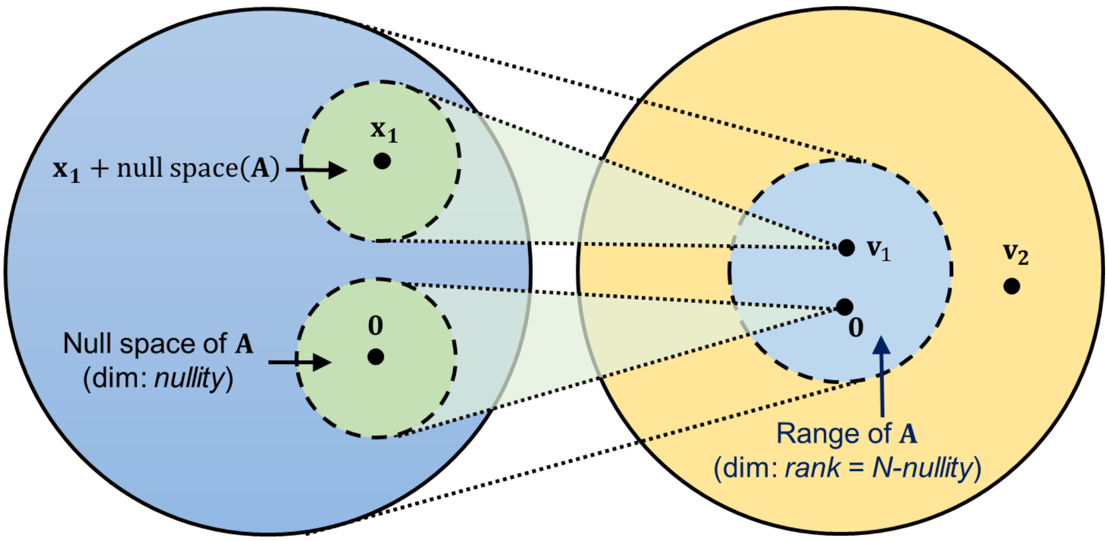
For a non-singular matrix, $\text{Null space}(\mathbf{A}) = \mathbf{0}$, so solutions $\mathbf{Ax} = \mathbf{v}$ are unique.

Every input is mapped to a single output. That is why there exists an inverse.



Singular square matrix ($\det(\mathbf{A}) = 0$):

For a singular matrix, full subspaces of the input are squashed into a single output



$\mathbf{Ax} = \mathbf{v_1} \rightarrow$ Solution: $\mathbf{x} = \mathbf{x}_1 + \text{null space}(\mathbf{A})$ (*infinite solutions*)
$\mathbf{Ax} = \mathbf{v_2} \rightarrow$ *No solution* for $\mathbf{x}$ (because $\mathbf{v_2}$ is outside the range)

5) Projection of 2D space into the x-axis:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

State the range, null space, rank and nullity of this transformation. Check that the theorem of dimensions is fulfilled.

Hence, Solve $\mathbf{v} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $\mathbf{x}$.

Range: the span of the columns: since the second column is zero it does not contribute to the span.

$$\text{range}(\mathbf{A}) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

Alternatively, we can write it in vector parametric form: $\text{range}(\mathbf{A}) = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Rank is the dimension of the range:

$$\text{rank}(\mathbf{A}) = 1$$

Null space: Solve the matrix equation $\mathbf{A}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1 + 0x_2 = 0 \\ 0x_1 + 0x_2 = 0 \end{cases}$$

x_2 is a free variable. The general solution is: $x_2 = \lambda$ and $x_1 = 0$, which we can write in parametric form as $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So that:

$$\text{null space}(\mathbf{A}) = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

Nullity is the dimension of the null space:

$$\text{nullity}(\mathbf{A}) = 1$$

The theorem of dimensions is fulfilled: Input space dimension = 2; rank = 1; nullity = 1.

a) Solve $\mathbf{A}\mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1 + 0x_2 = 3 \\ 0x_1 + 0x_2 = 0 \end{cases} \rightarrow \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \underbrace{\lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{Null space}}$$

as expected, the null-space can be added to any valid solution.

b) Solve $\mathbf{A}\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is outside the range, so the system does not have a solution.

6) Projection of 3D space into the x-axis:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \det \mathbf{A} = 0$$

State the range, null space, rank and nullity of this transformation. Check that the theorem of dimensions is fulfilled.

Hence, solve $\mathbf{v} = \mathbf{Ax} = (3,0,0)^T$ for $\mathbf{x}$.

Range: the span of the columns: since the second and third columns are zero they do not contribute to the span.

$$\mathrm{range}(\mathbf{A}) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

Alternatively, we can write it in vector parametric form: $\mathrm{range}(\mathbf{A}) = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Rank is the dimension of the range:

$$\mathrm{rank}(\mathbf{A}) = 1$$

Null space: Solve the matrix equation $\mathbf{Ax} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} x_1 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

x_2 and x_3 are free variables. The general solution is: $x_2 = \lambda, x_3 = \mu$ and $x_1 = 0$, which we can write in parametric form as:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So that:

$$\text{null space}(\mathbf{A}) = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

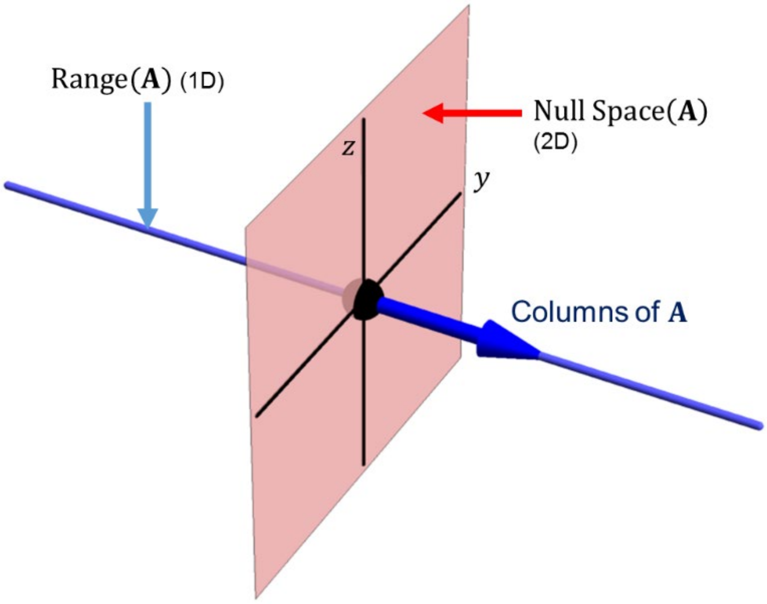
Nullity is the dimension of the null space:

$$\mathrm{nullity}(\mathbf{A}) = 2$$

The theorem of dimensions is fulfilled: Input space dimension = 3; rank = 1; nullity = 2.

Projection of 3D space onto the x-axis:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

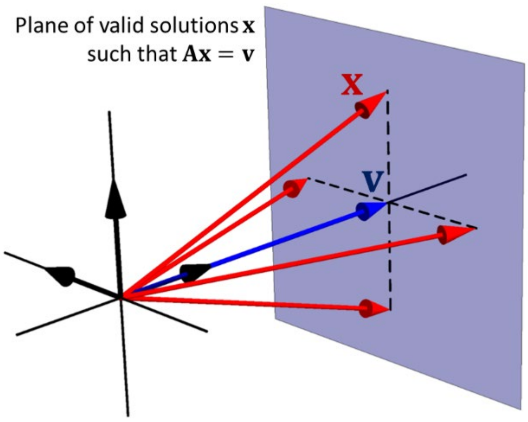


a) Solve $\mathbf{Ax} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = 3 \\ 0 = 0 \end{cases} \rightarrow$ x_2 and x_3 are free parameters.

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \underbrace{\lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{null space}}$$

as expected, the null-space can be added to any valid solution.



7) Projection of 3D space into $z = 0$ plane:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \det \mathbf{A} = 0$$

State the range, null space, rank and nullity of this transformation. Check that the theorem of dimensions is fulfilled.

Hence, solve $\mathbf{v} = \mathbf{A}\mathbf{x} = (3, -2, 0)^T$ for $\mathbf{x}$.

Range: the span of the columns: since the third column is zero only the first two contribute to the span.

$$\text{range}(\mathbf{A}) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Alternatively, we can write it in vector parametric form: $\text{range}(\mathbf{A}) = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Rank is the dimension of the range:

$$\text{rank}(\mathbf{A}) = 2$$

Null space: Solve the matrix equation $\mathbf{A}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x_1 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases}$$

x_3 is a free variable. The general solution is: $x_3 = \lambda, x_2 = 0$ and $x_1 = 0$, which we can write in parametric form as:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So that:

$$\text{null space}(\mathbf{A}) = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

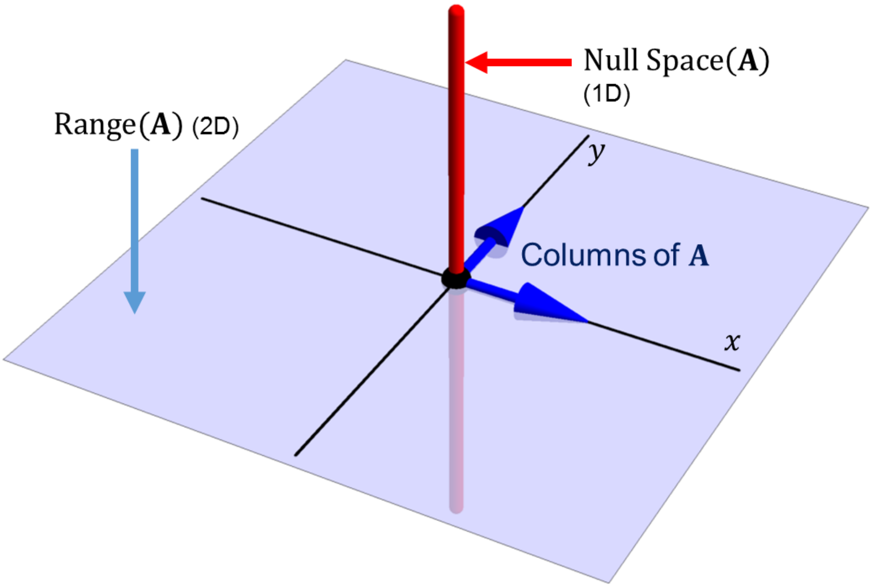
Nullity is the dimension of the null space:

$$\text{nullity}(\mathbf{A}) = 1$$

The theorem of dimensions is fulfilled: Input space dimension = 3; rank = 2; nullity = 1.

Visually we can understand: Projection of 3D space onto the plane $z = 0$:

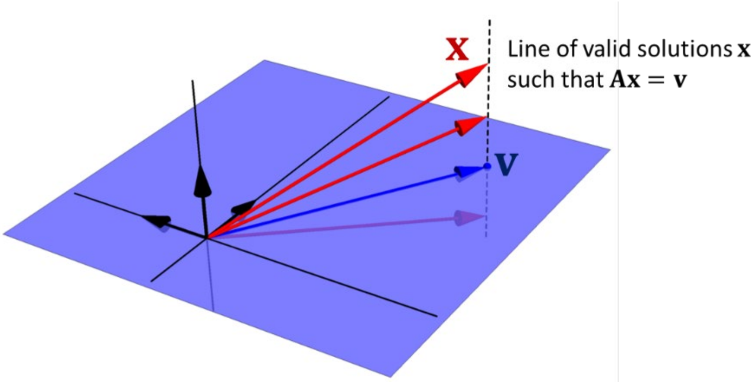
$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



a) Solve $\mathbf{Ax} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = 3 \\ x_2 = -2 \\ 0 = 0 \end{cases} \rightarrow \quad x_3 \text{ is a free parameter.}$$

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \underbrace{\lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{null space}}$$



as expected, the null-space can be added to any valid solution.

8) Problem: Consider the transformation given by the matrix $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$. Write down the input space, range and the null space of this matrix, state their dimensions, and check the theorem of dimensions. Draw all subspaces in the 2D plane. Solve $\mathbf{Ax} = \mathbf{v}$ for the cases:

(a) $\mathbf{v} = (-1,2)^T$
(b) $\mathbf{v} = (-1,-2)^T$

Solution:

<u>The input space</u> of this transformation is 2D space, with dimensions $N = 2$.

<u>The range</u>, or column space, of this matrix is the subspace of all possible outputs of this transformation and is given by the span of its columns. It is evident that the two column vectors are parallel, so they are linearly dependent:

$$\mathrm{range}(\mathbf{A}) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \end{pmatrix}\right\} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} \ \rightarrow\ \mathrm{rank}(\mathbf{A}) = \dim[\mathrm{range}(\mathbf{A})] = 1$$

The range of $\mathbf{A}$ is therefore the one-dimensional line given by $\mathbf{x} = \lambda(1,2)^T$. **The transformation compresses the entire 2D space into this line!** Indeed $\det(\mathbf{A}) = 0$.

<u>The null space</u> of this matrix is the subspace of all possible solutions to the equation $\mathbf{Ax} = \mathbf{0}$.

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This can be written as a system of equations:

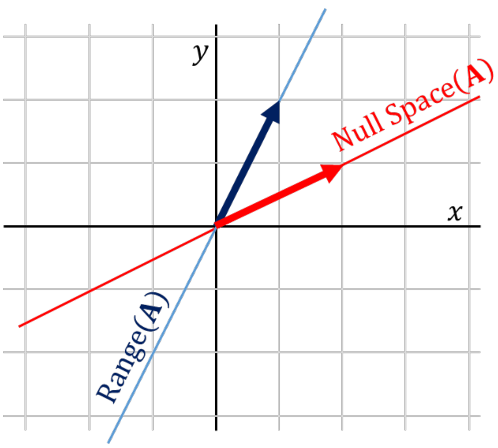
$$\begin{cases} x_1 - 2x_2 = 0 \\ 2x_1 - 4x_2 = 0 \end{cases}$$

Subtracting twice the first from the second leads to $0 = 0$, therefore we have one degree of freedom in the solution: $x_2 = \lambda$ and $x_1 = 2\lambda$. Writing this in vector form, the solution is:

$$\mathbf{x} = \lambda\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This is the subspace of solutions to $\mathbf{Ax} = \mathbf{0}$, and therefore is the null space of the matrix.

$$\text{null space}(\mathbf{A}) = \mathrm{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \ \rightarrow\ \mathrm{nullity}(\mathbf{A}) = \dim[\text{null space}(\mathbf{A})] = 1$$

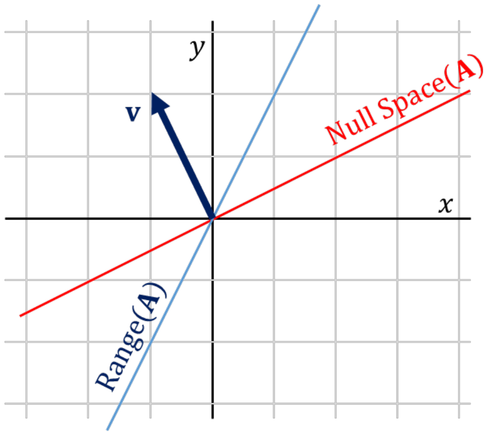


The null space tells us about the direction in which the space is being “squashed” into the origin, and it will help us a lot for solving the inverse problem.

The theorem of dimensions is fulfilled, as $\mathrm{rank}(\mathbf{A}) = \mathrm{N} - \mathrm{nullity}(\mathbf{A})$.

a) Solve $\mathbf{Ax} = (-1,2)^{\boldsymbol{T}}$

This value of $\mathbf{v}$ falls outside the range of $\mathbf{A}$, therefore there are no solutions!



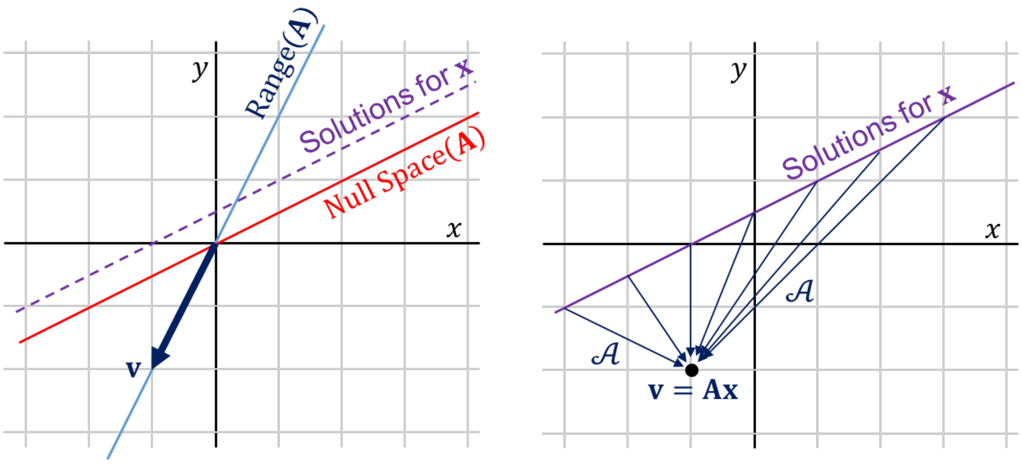
b) Solve $\mathbf{Ax} = (-1,-2)^{\boldsymbol{T}}$

This value of $\mathbf{v}$ falls inside the range of $\mathbf{A}$, so we can find at least one solution, and then add the null space as a degree of freedom.

$$\begin{pmatrix}1 & -2\\2 & -4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix} \rightarrow \begin{cases}x_1 - 2x_2 = -1\\2x_1 - 4x_2 = -2\end{cases}$$

We know we have one degree of freedom, so take $x_2 = \lambda$, therefore we can solve $x_1 = -1 + 2\lambda$.

In vector form: $\mathbf{x} = \begin{pmatrix}-1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\1\end{pmatrix}$



It is very difficult to interpret this transformation in "visual" terms, because it is not a projection. However, it does squish whole lines (always parallel to the null space) into single points somewhere along the line of the range.

D. NULL SPACE AND RANGE USING GAUSSIAN ELIMINATION

The simplest and most efficient way of calculating the range and null-space of matrices is by using **Gaussian Elimination**. This technique is thoroughly explained in the next chapter. Once you learn it, you can come back here to do these problems:

NULL SPACE:

To obtain the null space we can, by definition, simply solve the system $\mathbf{Ax} = \mathbf{0}$.

$\text{Null space}\{\mathbf{A}\} =$ General solution to system $\mathbf{Ax} = \mathbf{0}$

Gaussian elimination will tell us which variables are free (dimensions of the null space). The free variables correspond to the columns with no pivot in row echelon form.

$\text{Nullity}\{\mathbf{A}\} =$ Number of columns without pivot in row echelon form $\mathbf{U}$

RANGE:

To obtain the range we need to find the span of all the columns. For this we need to check if the columns are linearly dependent or independent.

Columns of $\mathbf{A}$ $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ are linearly independent $\Leftrightarrow$ $\sum x_i \mathbf{v}_i = 0$ Only when all $x_i = 0$ $\Leftrightarrow$ $\mathbf{Ax} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$

Gaussian elimination of $\mathbf{Ax} = \mathbf{0}$ will tell us which columns of the original matrix where linearly dependent on the others (columns with no pivot). The range will be equal to the span of the columns in the original matrix whose corresponding columns in row echelon form had pivots:

$\text{Range}\{\mathbf{A}\} =$ span of the columns in $\mathbf{A}$ that have pivot in the row echelon form $\mathbf{U}$

$\text{Rank}\{\mathbf{A}\} =$ Number of columns with pivot in row echelon form $\mathbf{U}$

9) Find the range, rank, null space and nullity of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

Solution:

First, let's find the **null space** of $\mathbf{A}$ by solving $\mathbf{Ax} = \mathbf{0}$.

The augment matrix for the system $\mathbf{Ax} = \mathbf{0}$ is written as:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 2 & -4 & 2 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right)$$

Now we perform Gaussian elimination to reduce this matrix into row echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 2 & -4 & 2 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right) \xrightarrow[R_3+R_1\to R_3]{R_2-2R_1\to R_2} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The first column and third columns have pivot. The second column has no pivot and thus corresponds to free variables. The general solution, by inverse substitution, is therefore:

$$x_2 = \alpha$$

$$x_3 = 0$$

$$x_1 - 2(\alpha) - (0) = 0 \;\rightarrow\; x_1 = 2\alpha$$

Which can be written as:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Therefore:

$$\text{null space}\{\mathbf{A}\} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

a one-dimensional space. The nullity, being the dimension of the null space, is therefore:

$$\text{nullity}\{\mathbf{A}\} = 1$$

Now, let's find the range. The range is equal to the span of the columns. But the columns which had no pivots after gaussian elimination do not contribute to the span of the columns. So, we can take the first and third column in the original matrix:

In summary, **the range is always given by the span of the columns in the original matrix corresponding to the columns that have pivots in the row-echelon form**:

$$\text{range}\{\mathbf{A}\} = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\}$$

Which is a two-dimensional subspace, hence:

$$\text{rank}\{\mathbf{A}\} = 2$$

The theorem of dimensions is fulfilled. Input dimension $N = 3$. Rank $= 2$. Nullity$= 1$.

10) Find the range, rank, null space and nullity of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & -4 & -2 \\ 1 & -2 & -1 \\ -2 & 4 & 2 \end{pmatrix}$$

Solution:

First, let's find the **null space** of $\mathbf{A}$ by solving $\mathbf{Ax} = \mathbf{0}$.

The augment matrix for the system $\mathbf{Ax} = \mathbf{0}$ is written as:

$$\left(\begin{array}{ccc|c} 2 & -4 & -2 & 0 \\ 1 & -2 & -1 & 0 \\ -2 & 4 & 2 & 0 \end{array}\right)$$

Now we perform Gaussian elimination to reduce this matrix into row echelon form:

$$\left(\begin{array}{ccc|c} 2 & -4 & -2 & 0 \\ 1 & -2 & -1 & 0 \\ -2 & 4 & 2 & 0 \end{array}\right) \xrightarrow[R_3+R_1\to R_3]{R_2-\left(\frac{1}{2}\right)R_1\to R_2} \left(\begin{array}{ccc|c} 2 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The first column has pivot. The two last columns have no pivot and thus correspond to free variables. The general solution, by inverse substitution, is therefore:

$$x_3 = \alpha$$

$$x_2 = \beta$$

$$2x_1 - 4\beta - 2\alpha = 0 \;\rightarrow\; \quad x_1 = \alpha + 2\beta$$

Which can be written as:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Therefore:

$$\text{null space}\{\mathbf{A}\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right\}$$

a two-dimensional space. The nullity, being the dimension of the null space, is therefore:

$$\text{nullity}\{\mathbf{A}\} = 2$$

Now, let's find the range. The range is equal to the span of the columns. But the columns which had no pivots after gaussian elimination do not contribute to the span of the columns.

In summary, **the range is always given by the span of the columns in the original matrix corresponding to the columns that had pivots in the row-echelon form**:

$$\text{range}\{\mathbf{A}\} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right\}$$

Which is a one-dimensional line, hence:

$$\text{rank}\{\mathbf{A}\} = 1$$

The theorem of dimensions is fulfilled. Input dimension $N = 3$. Rank $= 1$. Nullity$= 2$.

11) Find the range, rank, null space and nullity of the following matrix.

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix}$$

and solve the system $\mathbf{Ax} = (4,0,-7)^T$.

Solution:

First, let's find the **null space** of $\mathbf{A}$ by solving $\mathbf{Ax} = \mathbf{0}$.

The augment matrix for the system $\mathbf{Ax} = \mathbf{0}$ is written as:

$$\left(\begin{array}{ccc|c} 2 & 4 & 3 & 0 \\ 1 & -2 & -2 & 0 \\ -3 & 3 & 2 & 0 \end{array}\right)$$

Now we perform Gaussian elimination to reduce this matrix into row echelon form:

$$\left(\begin{array}{ccc|c} 2 & 4 & 3 & 0 \\ 1 & -2 & -2 & 0 \\ -3 & 3 & 2 & 0 \end{array}\right) \xrightarrow[R_3+\left(\frac{3}{2}\right)R_1\to R_3]{R_2-\left(\frac{1}{2}\right)R_1\to R_2} \left(\begin{array}{ccc|c} 2 & 4 & 3 & 0 \\ 0 & -4 & -7/2 & 0 \\ 0 & 9 & 13/2 & 0 \end{array}\right)$$

Let's avoid fractions (this is optional but helps avoiding mistakes) by multiplying rows:

$$\xrightarrow[2R_3\to R_3]{2R_2\to R_2} \left(\begin{array}{ccc|c} 2 & 4 & 3 & 0 \\ 0 & -8 & -7 & 0 \\ 0 & 18 & 13 & 0 \end{array}\right)$$

And now we continue with the steps to reduce into row echelon form:

$$\xrightarrow{R_3+\left(\frac{9}{4}\right)R_2\to R_3} \left(\begin{array}{ccc|c} 2 & 4 & 3 & 0 \\ 0 & -8 & -7 & 0 \\ 0 & 0 & -11/4 & 0 \end{array}\right)$$

All the columns have pivots! Therefore, there are no free variables. Therefore the null space has no dimensions (it is only the origin). We can check this by doing inverse substitution and finding the general solution: $x_1 = x_2 = x_3 = 0$.

Therefore:

$$\text{null space}\{\mathbf{A}\} = \mathbf{0}$$

a zero-dimensional space. The nullity, being the dimension of the null space, is therefore zero:

$$\text{nullity}\{\mathbf{A}\} = 0$$

Now, let's find the range. Since all columns had pivots after gaussian elimination, we know that all the columns are linearly independent. The matrix has full-rank, and hence its range is simply the three dimensional span of its three linearly independent columns.

In summary, **the range is always given by the span of the columns in the original matrix corresponding to the columns that had pivots in the row-echelon form**:

$$\text{range}\{\mathbf{A}\} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}\right\}$$

Which is a three-dimensional space, hence:

$$\text{rank}\{\mathbf{A}\} = 3$$

The theorem of dimensions is fulfilled. Input dimension $N = 3$. Rank $= 3$. Nullity $= 0$.

This matrix is full-rank, which means it has an inverse. We could use the inverse to solve the system $\mathbf{Ax} = (4,0,-7)^T$. Alternatively, we can use gaussian elimination (we can reuse the same gaussian steps for the matrix! We just need to apply the steps to the independent term)

$$\left(\begin{array}{ccc|c} 2 & 4 & 3 & 4 \\ 1 & -2 & -2 & 0 \\ -3 & 3 & 2 & -7 \end{array}\right) \xrightarrow[R_3+\left(\frac{3}{2}\right)R_1 \to R_3]{R_2-\left(\frac{1}{2}\right)R_1 \to R_2} \left(\begin{array}{ccc|c} 2 & 4 & 3 & 4 \\ 0 & -4 & -7/2 & -2 \\ 0 & 9 & 13/2 & -1 \end{array}\right) \xrightarrow[2R_3 \to R_3]{2R_2 \to R_2} \left(\begin{array}{ccc|c} 2 & 4 & 3 & 4 \\ 0 & -8 & -7 & -4 \\ 0 & 18 & 13 & -2 \end{array}\right)$$

$$\xrightarrow{R_3+\left(\frac{9}{4}\right)R_2 \to R_3} \left(\begin{array}{ccc|c} 2 & 4 & 3 & 4 \\ 0 & -8 & -7 & -4 \\ 0 & 0 & -11/4 & -11 \end{array}\right)$$

Hence, solving by inverse substitution:

3rd row: $-11x_3 = -44 \to \ x_3 = 4$

2nd row: $-8x_2 - 7(4) = -4 \ \to x_2 = \frac{24}{-8} = -3$

1st row: $2x_1 + 4(-3) + 3(4) = 4 \ \to \ x_1 = 2$

12) Find the range, rank, null space and nullity of the following non-rectangular matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{pmatrix}$$

Solution: This case is good practice because it is a non-rectangular matrix. Its input is 4-dimensional, while its output is 3-dimensional. Still, all the theory applies. The null space is a subspace of the input space which is mapped to zero in the output. The range is a subspace of the output space into which all the input space is mapped.

First, let's find the **null space** of $\mathbf{A}$ by solving $\mathbf{Ax} = \mathbf{0}$.

The augment matrix for the system $\mathbf{Ax} = \mathbf{0}$ is written as:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 2 & 0 & 2 & 2 & 0 \\ 4 & 1 & 3 & 1 & 0 \end{array}\right)$$

Now we perform Gaussian elimination to reduce this matrix into row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 2 & 0 & 2 & 2 & 0 \\ 4 & 1 & 3 & 1 & 0 \end{array}\right) \xrightarrow[R_3-4R_1 \to R_3]{R_2-2R_1 \to R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 2 & 6 & 0 \\ 0 & -3 & 3 & 9 & 0 \end{array}\right) \xrightarrow{R_3-\left(\frac{3}{2}\right)R_2 \to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now we make our lives easier by dividing some rows by integer numbers to make numbers smaller:

$$\xrightarrow{\left(\frac{1}{2}\right)R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & -1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now we can solve the system by inverse substitution. The third and fourth column have no pivots, so they are free variables. The first two columns have pivot, so they are linearly independent columns.

$$x_3 = \alpha \text{ and } x_4 = \beta$$

$$\text{2nd row: } -x_2 + \alpha + 3\beta = 0 \ \to \ x_2 = \alpha + 3\beta$$

$$\text{1st row: } x_1 + (\alpha + 3\beta) - 2\beta = 0 \to x_1 = -\alpha - \beta$$

So, the general solution to $\mathbf{Ax} = \mathbf{0}$ is:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

Which means that:

$$\text{null space}\{\mathbf{A}\} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This is a two-dimensional plane living in the input 4-dimensional space, therefore, the nullity, being the dimension of the null space:

$$\text{nullity}\{\mathbf{A}\} = 2$$

Next, we must find the range of $\mathbf{A}$. The range is the span of the columns of $\mathbf{A}$. However, some columns might not be contributing to the span (being linearly dependent on others). We can find out a set of columns that are linearly dependent thanks to Gaussian elimination. We saw that the third and fourth columns had no pivot, therefore, the 3rd and 4th columns can be written as a linear combination of the other two. The first two columns did have a pivot, therefore, they are linearly independent.

In summary, we can find the range of $\mathbf{A}$ by considering the **span of the columns in the original matrix which had a pivot in the row echelon form after gaussian elimination:**

$$\text{range}\{\mathbf{A}\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This is a two-dimensional subspace of the output three-dimensional space. The dimension of the range is two, hence:

$$\text{rank}\{\mathbf{A}\} = 2$$

The theorem of dimensions is fulfilled. Input dimension $N = 4$. Rank $= 2$. Nullity$= 2$.

13) Find the range, rank, null space and nullity of the linear operation corresponding to differentiation with respect to x in the vector space of polynomials of degree equal or smaller than 4, with the basis $\{1, x, x^2, x^3, x^4\}$. Check that the theorem of dimensions is obeyed.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

This matrix was derived in a previous chapter. Its size is 4×5 because it converts polynomials of degree 4 (i.e. with 5 dimensions) into polynomials of degree 3 (i.e. with 4 dimensions).

Solution:

Remember that we are associating polynomials with vectors:

$$\mathbf{p} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \to p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

First, let's find the **null space** by solving $\mathbf{Ap} = \mathbf{0}$. This will also allow us to find out which columns are linearly independent (the columns without pivot).

$$\mathbf{Ap} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \mathbf{0}$$

Written as an augmented matrix, we have:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{array}\right)$$

Now we would perform gaussian elimination to bring it into row echelon form. Fortunately, no steps are required because the matrix is already in row-echelon form, so we can directly read all the information from it. The first column does not have a pivot, and so is a free variable, and so it is linearly dependent on the other columns (this was obvious! It is all zeroes, so any other column scaled by zero results in this column!). The last 4 columns are linearly independent, because they all have pivots.

By giving the first variable (no pivot) a free parameter $a_0 = \lambda$ we can solve the others by inverse substitution. The solution is trivial: $a_1 = a_2 = a_3 = a_4 = 0$. Therefore, the null space of $\mathbf{A}$ is given by:

$$\text{null space}(\mathbf{A}) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Notice that the null space is a line living in the input 5-dimensional space. The nullity is equal to the dimensions of this space. Therefore, **nullity** of $\mathbf{A}$ is 1.

The **range of the matrix** is the **subspace given by the span of the columns**. It is a subspace of the output space of the matrix, i.e. the range exists in the 4-dimensional output space.

We know that the last 4 columns are linearly independent (because they had a pivot after Gaussian elimination), and the first column is linearly dependent on the others (because it didn't have a pivot after Gaussian elimination), therefore we can write the range of the matrix as the span of the last four columns:

$$\mathrm{range}(\mathbf{A}) = \mathrm{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\} = \alpha\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\delta\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

This is a four-dimensional subspace, so its dimension is 4. The **rank** of $\mathbf{A}$ is 4. Interestingly, the range of $\mathbf{A}$ (non-square matrix) in this case is equal to the entire output 4-dimensional space.

The theorem of dimensions is fulfilled. Input dimension $N = 5$. Rank $= 4$. Nullity$= 1$.

It is instructive to translate all these results into the language of polynomials (where it all becomes quite trivial).

$$\text{null space}\left(\frac{\mathrm{d}}{\mathrm{d}x}(4^{\text{th}}\text{ order polynomials})\right) = \mathrm{span}\{1\} = a_0$$

$$\text{range}\left(\frac{\mathrm{d}}{\mathrm{d}x}(4^{\text{th}}\text{ order polynomials})\right) = \mathrm{span}\{1, x, x^2, x^3\} = b_0 + b_1x + b_2x^2 + b_3x^3$$

14) Following the previous problem. Solve the indefinite integration of polynomials (which is the inverse of differentiation) by using vector notation and linear algebra techniques:

$$p(x) = \int (5 + x^2)\mathrm{d}x \;\; \rightarrow \;\; \frac{\mathrm{d}}{\mathrm{d}x}(p(x)) = 5 + x^2$$

In the language of vectors, we are trying to solve:

$$\frac{\mathrm{d}}{\mathrm{d}x}(p(x)) = 5 + x^2 \quad \rightarrow \quad \mathbf{Ap} = \begin{pmatrix}5\\0\\1\\0\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&4\end{pmatrix}\begin{pmatrix}a_0\\a_1\\a_2\\a_3\\a_4\end{pmatrix} = \begin{pmatrix}5\\0\\1\\0\end{pmatrix}$$

The augment matrix is already in row-echelon form, so we can solve it by inverse substitution:

$$\left(\begin{array}{ccccc|c}0&1&0&0&0&5\\0&0&2&0&0&0\\0&0&0&3&0&1\\0&0&0&0&4&0\end{array}\right)$$

$a_0 = c$ is a free parameter (null space); $a_1 = 5$; $a_2 = 0$; $a_3 = 1/3$; $a_4 = 0$.

The general solution is therefore, in vector language, a line in 5-dimensional space:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 1/3 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Which, translated to polynomial language, gives us the familiar:

$$p(x) = 5x + \frac{1}{3}x^3 + c$$

This is an explanation, in linear algebra terms, for why an indefinite integral (the inverse of the derivative) always has an arbitrary integration constant c! The linear transformation has a zero determinant, and so we must add the null space to any solution!

Extra note: This also applies nicely to linear ordinary differential equations which can also be seen as a linear transformation.

$$2\frac{\mathrm{d}^2}{\mathrm{d}x^2}p(x) - \frac{\mathrm{d}}{\mathrm{d}x}p(x) + 3p(x) = x^2$$

$$\left(2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x} + 3\right)p(x) = x^2$$

$$\mathcal{A}\big(p(x)\big) = x^2$$

$$\mathbf{Ap} = \mathbf{v}$$

You know from your study of differential equations, that the general solution is equal to the particular solution plus the homogeneous solution. The homogeneous solution is the solution to the system when the independent coefficients are zero, i.e. the homogeneous solution is the null space!

$$\mathbf{p} = \mathbf{p}_P + \mathbf{p}_H = \mathbf{p}_P + \text{null space}\{\mathbf{A}\}$$

This is not an analogy or a coincidence. This is an exact equivalence. That is why some differential equations are called LINEAR differential equations and always fulfil the above.

2.4 LINEAR SYSTEMS OF EQUATIONS AND GAUSSIAN ELIMINATION

Linear systems of equations are used very often in science and engineering when solving linear laws of physics, and equally often when simplifying non-linear equations using a linear approximation.

A. LINEAR SYSTEM OF EQUATIONS AS A MATRIX-VECTOR MULTIPLICATION

A linear system of equations is a collection of M equations and N unknowns, in which the **unknowns** $(x_1, x_2, \cdots, x_N)$ appear as linear combinations with simple scaling **coefficients** (a_{ij}) and each equation has **independent terms** (b_i). We can always write a linear system of equations as follows:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots a_{1N}x_N = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2N}x_N = b_2 \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots a_{MN}x_N = b_M \end{cases}$$

This system can be written and interpreted as a simple matrix-vector multiplication

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix}}_{\mathbf{b}}$$

This means that, in order to solve the values of the variables, we need to find a vector $\mathbf{x}$, which when transformed according to matrix $\mathbf{A}$ results in a vector $\mathbf{b}$. In other words, we need to solve the inverse problem $\mathbf{Ax} = \mathbf{b}$. This means we can apply the results of the previous section.

- When $\det(\mathbf{A}) \neq 0$, the system has a unique solution given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
- When $\det(\mathbf{A}) = 0$, the system may have:
    - no solutions (when $\mathbf{b}$ is outside the range of $\mathbf{A}$)
    - infinite solutions with Z degrees of freedom (when $\mathbf{b}$ is inside the range of $\mathbf{A}$).
        - $Z = \mathrm{nullity}(\mathbf{A})$

In practice, calculating the inverse $\mathbf{A}^{-1}$ using the general recipe which involved determinants is a very long procedure, even for a computer (for high number of dimensions). In real-life, the inverse is never used. Instead, a much faster algorithm is used, called Gaussian elimination.

B. GAUSSIAN ELIMINATION

Gaussian elimination is a very efficient and general way to solve a system of linear equations in all possible cases. It works for unique solutions, no solutions and infinite solutions. This method can also be used to find the rank of a matrix, the range of a matrix, the determinant of a matrix, and to calculate the inverse of an invertible square matrix. It is much faster than the usual methods.

Using Gaussian elimination, computers can solve systems of equations with tens of thousands of variables and equations.

To start solving the system $\mathbf{Ax} = \mathbf{b}$, we write the matrix $\mathbf{A}$ next to the vector $\mathbf{b}$ in the same matrix, simply for convenience. This is called the **augmented matrix** $\mathbf{A}|\mathbf{b}$:

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} & b_M \end{array}\right)$$

Gaussian elimination performs operations called "gaussian steps" on this augmented matrix until it reaches **row echelon form**. A matrix is in row echelon form if:

- All zero rows are at the bottom of the matrix
- The first nonzero number of each row (called the **pivot**) is always to the right of the pivot of the row above. Therefore, **every pivot has all zeroes below and to the left of it**.

Example of a matrix in row echelon form:

$$\left(\begin{array}{cccccccc|c} a & * & * & * & * & * & * & * & * \\ 0 & 0 & b & * & * & * & * & * & * \\ 0 & 0 & 0 & c & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & d & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & e & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array}\right)$$

Columns with no pivots

- a, b, c, d, e are the **pivots** (**leftmost non-zero numbers in each row**)
- asterisks $*$ are any number (non-zero or zero)

A matrix that has been reduced to this form tells you all its secrets if you know how to read them:

- Row echelon form is triangular, so its determinant is the product of the diagonal elements.
- **Columns with pivot correspond to linearly independent vectors**, while columns with no pivot are linearly dependent on those with pivot.
    - The range of the original matrix (before gaussian elimination) can be obtained as the span of the columns in the original matrix which have pivot in the row echelon form.
    - The number of columns with pivot is therefore equal to $\mathrm{rank}(\mathbf{A})$.
    - If all columns have a pivot, all columns are linearly independent and the original matrix has full rank and is non-singular.
- **Columns without pivot correspond to linearly dependent vectors**, linear combination of those with pivot
    - the unknown variable corresponding to columns with no pivot will be free variables (degree of freedom of the solution)

SOLVING THE SYSTEM AFTER GAUSSIAN ELIMINATION:

After Gaussian elimination, the resulting system of equations is equivalent:

$$(\mathbf{A}|\mathbf{b}) \xrightarrow{\text{Gaussian elimination}} (\mathbf{U}|\mathbf{c})$$

This means that **the system $\mathbf{Ux} = \mathbf{c}$ has the same solutions as the original system $\mathbf{Ax} = \mathbf{b}$.**

Different cases arise depending on the form of the row echelon form system $(\mathbf{U}|\mathbf{c})$:

Case 1: **Unique solution exists**. $\mathbf{U}$ is an upper triangular matrix (all columns have pivot).

$$\left(\begin{array}{cccc|c} \neq 0 & * & * & * & * \\ 0 & \neq 0 & * & * & * \\ 0 & 0 & \neq 0 & * & * \\ 0 & 0 & 0 & \neq 0 & * \end{array}\right)$$

Solve the system by inverse substitution: solve the last variable starting from the bottom row, which involves only one variable. Substitute the solution in the previous row, and so on.

Remember that the first column corresponds to the first variable x_1, second column to x_2, etc.

Case 2: **No solutions**. Some rows of $\mathbf{U}$ are all zero, but column $\mathbf{c}$ is not zero in at least one of them.

This means that $\mathbf{c}$ is outside the range of $\mathbf{U}$, and therefore $\mathbf{x}$ is outside the range of $\mathbf{A}$, so **the system has no solution**. Note this case never happens when solving the homogeneous system $\mathbf{Ax} = \mathbf{0}$, because $\mathbf{0}$ is always inside the range of every matrix.

$$\left(\begin{array}{cccc|c} \neq 0 & * & * & * & * \\ 0 & \neq 0 & * & * & * \\ 0 & 0 & 0 & 0 & \neq 0 \\ 0 & 0 & 0 & 0 & * \end{array}\right)$$

Case 3: **Infinite solutions**. Rest of cases.

i.e. all the zero rows of $\mathbf{U}$ are also zero for $\mathbf{c}$, and $\mathbf{U}$ has columns with no pivot (in other words, after removing all zero rows, $\mathbf{U}$ has more columns than rows): the system then has infinite solutions.

$$\begin{array}{c} \left(\begin{array}{cccccc|c} \neq 0 & * & * & * & * & * & * \\ 0 & \neq 0 & * & * & * & * & * \\ 0 & 0 & 0 & \neq 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \neq 0 & * \end{array}\right) \\ \begin{array}{cccccc} & & \uparrow\text{No pivot} & & \uparrow\text{No pivot (free variables)} & \end{array} \end{array}$$

Solve the system by assigning free parameters $\lambda, \mu, \cdots$ to the variables corresponding to the columns with no pivot. Solve the other variables by inverse substitution.

THE GAUSSIAN STEP:

How do you get the augmented matrix into row echelon form?

Remember when we solved a system of equations $\begin{cases} x + y = 2 \\ -x + y = 3 \end{cases}$ we could do linear combinations of the equations in order to isolate the variables. Now we will do similar things:

There are three types of gaussian steps:

- $R_i + \lambda R_j \rightarrow R_i$: Get any row and add to it a scaled version of another row (scalars λ can be negative, i.e. subtraction of rows).
- $\lambda R_i \rightarrow R_i$: Multiply a row by a non-zero scalar.
- $R_3 \leftrightarrow R_2$: Swap any two rows.

Example:

$$\begin{pmatrix} 1 & 1 & 2 \\ -2 & -2 & -3 \\ 4 & 6 & 8 \end{pmatrix} \xrightarrow{R_2 + 2R_1 \rightarrow R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 4 & 6 & 8 \end{pmatrix} \xrightarrow{R_3 - 4R_1 \rightarrow R_3} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $R_2 + 2R_1 \rightarrow R_2$: second row plus two times the first row, placed in second row.
- $R_3 - 4R_1 \rightarrow R_3$: third row minus four times first row, placed in third row.
- $R_3 \leftrightarrow R_2$: swap row 3 with row 2.

Gaussian steps ensure that the augmented matrix in every step corresponds to an equivalent system of equations, that is, the solutions are the same. The gaussian step also preserves the linear dependence or independence of each individual column.

Gaussian steps allow turning any matrix into row echelon form:

- Use the pivot of the first row to turn every non-zero coefficient below it to zero
- Move onto the next row. If any row below it has a pivot to the left of the current row's pivot, swap the rows (the aim is that no row has any pivot to the left of the previous rows). Use this new pivot to turn every non-zero coefficient below it to zero.

1) Example: Solve the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 - 4 = 0 \\ 2x_1 + x_3 + x_4 = 6 - 2x_2 \\ 3x_1 + x_2 = 7 - 2x_3 - x_4 \\ x_1 + x_2 + 2(2x_3 + x_4 - 4) = 0 \end{cases}$$

First write the system putting all variables on the left, and all independent terms on the right:

$$\begin{cases} 1x_1 + 1x_2 + 1x_3 + 1x_4 = 4 \\ 2x_1 + 2x_2 + 1x_3 + 1x_4 = 6 \\ 3x_1 + 1x_2 + 2x_3 + 1x_4 = 7 \\ 1x_1 + 1x_2 + 4x_3 + 2x_4 = 8 \end{cases}$$

Write the system in matrix form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 1 & 1 & 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 7 \\ 8 \end{pmatrix}$$

Write $\mathbf{Ax} = \mathbf{b}$ as an augmented matrix $(\mathbf{A}|\mathbf{b})$, and start performing Gaussian steps to bring the matrix into row echelon form.

1. Use the pivot to introduce zeroes below it by adding multiples of the first row to the others
2. Swap rows if necessary, to make sure that top rows have the leftmost pivots

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 1 & 6 \\ 3 & 1 & 2 & 1 & 7 \\ 1 & 1 & 4 & 2 & 8 \end{array}\right) \xrightarrow[R_4 - R_1 \to R_4]{\substack{R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -1 & -2 \\ 0 & -2 & -1 & -2 & -5 \\ 0 & 0 & 3 & 1 & 4 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -2 & -1 & -2 & -5 \\ 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 3 & 1 & 4 \end{array}\right)$$

$$\xrightarrow{R_4 + 3R_3 \to R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -2 & -1 & -2 & -5 \\ 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right)$$ in the form $(\mathbf{U}|\mathbf{c})$: Unique solution

Solve the **unique solution** by inverse substitution:

4[th] row: $-2x_4 = -2 \quad \to \quad x_4 = 1$
3[rd] row: $-x_3 - x_4 = -2 \quad \to \quad x_3 = 2 - x_4 = 1$
2[nd] row: $-2x_2 - x_3 - 2x_4 = -5 \quad \to \quad x_2 = (1/2)(5 - x_3 - 2x_4) = 1$
1[st] row: $x_1 + x_2 + x_3 + x_4 = 4 \to x_1 = 4 - x_2 - x_3 - x_4 = 1$

We can write the solution in compact vector form:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Also $\det \mathbf{A} = (-1) \det \mathbf{U}$ (due to the swap) $= (-1)(1)(-2)(-1)(-2) = 4$

2) Example: Solve the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 7 \\ x_1 + x_2 + x_4 = 5 \\ 2x_1 + 2x_2 + 3x_3 + x_4 = 10 \end{cases}$$

Write the system in matrix form (notice it is not a square matrix):

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 10 \end{pmatrix}$$

Write $\mathbf{Ax} = \mathbf{b}$ as an augmented matrix $(\mathbf{A}|\mathbf{b})$, and start performing Gaussian steps to bring the matrix into row echelon form.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 1 & 1 & 0 & 2 & 5 \\ 2 & 2 & 3 & 1 & 10 \end{array}\right) \xrightarrow[R_3-2R_1\to R_3]{R_2-R_1\to R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 & -4 \end{array}\right) \xrightarrow{R_3+R_2\to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right)$$

in row echelon form $(\mathbf{U}|\mathbf{c})$. This system is incompatible, **no solutions** exist, because the all-zeros rows of $\mathbf{U}$ are not zero in $\mathbf{c}$ too.

3) Example: Solve the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 7 \\ x_1 + x_2 + 2x_4 = 5 \\ 2x_1 + 2x_2 + 3x_3 + x_4 = 16 \end{cases}$$

Write the system in matrix form (notice it is not a square matrix):

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 16 \end{pmatrix}$$

Write $\mathbf{Ax} = \mathbf{b}$ as an augmented matrix $(\mathbf{A}|\mathbf{b})$, and start performing Gaussian steps to bring the matrix into row echelon form.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 1 & 1 & 0 & 2 & 5 \\ 2 & 2 & 3 & 1 & 16 \end{array}\right) \xrightarrow[R_3-2R_1\to R_3]{R_2-R_1\to R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right) \xrightarrow{R_3+R_2\to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

in row echelon form $(\mathbf{U}|\mathbf{c})$. This system has **infinite solutions** because, removing all-zero rows, it has more columns than rows. We solve it as follows:

First: Variables associated with no-pivot columns are given independent free parameters: $x_4 = \alpha$ and $x_2 = \beta$.

Then we solve the other variables by inverse substitution:
2nd row: $-x_3 + x_4 = -2 \quad \rightarrow \quad x_3 = 2 + x_4 = 2 + \alpha$

1st row: $x_1 + x_2 + x_3 + x_4 = 7 \;\rightarrow\; x_1 = 7 - x_2 - x_3 - x_4 = 7 - \beta - (2+\alpha) - \alpha = 5 - 2\alpha - \beta$

We can write the general solution in compact vector form as:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The general solution is a plane in four dimensions.

Note: For this example I took the previous example which had no solutions and modified the independent coefficients only, to make sure that $\mathbf{b}$ fell inside the column space of $\mathbf{A}$ so that there were solutions this time. This means that the steps for Gaussian elimination where the same as in the previous exercise. This means that once we perform the Gaussian steps for any system $\mathbf{Ax} = \mathbf{b_1}$, we can solve related systems $\mathbf{Ax} = \mathbf{b_2}, \mathbf{b_3}, \cdots$ by just remembering $\mathbf{U}$ and performing the gaussian steps on the different $\mathbf{b_i}$'s and doing inverse substitution.

4) Consider the transformation $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & -1 & -1 & 1 \\ 0 & 3 & -1 & 3 \end{pmatrix}$. Find the range and the null space of this transformation.

Solution:
To obtain the range we need to check if the columns are linearly dependent or independent. It turns out we can do that by solving a system $\mathbf{Ax} = \mathbf{0}$.

Columns of $\mathbf{A}$ $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ are linearly independent	$\Leftrightarrow$	$\sum x_i \mathbf{v}_i = 0$ Only when all $x_i = 0$	$\Leftrightarrow$	$\mathbf{Ax} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$

Therefore, we can solve the system $\mathbf{Ax} = \mathbf{0}$:

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & -1 & -1 & 1 \\ 0 & 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write down the augmented matrix and perform Gaussian steps:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 2 & -1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 3 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1 \to R_2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & -3 & 1 & -3 & 0 \\ 0 & 3 & -1 & 3 & 0 \end{array}\right) \xrightarrow{R_3 + R_2 \to R_3} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & -3 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

<u>**This tells us that the third and fourth columns (no pivots) of the original matrix where linearly dependent on the first and second column**</u>**. The range is equal to the span of the columns in the original matrix,** and we can discard the linearly dependent columns, which therefore leaves us with the span of the first and second columns (i.e. the span of the columns in $\mathbf{A}$ that have pivot in $\mathbf{U}$):

$$\text{Range}(\mathbf{A}) = \text{span}\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\3\end{pmatrix}\right\}$$

The rank of the matrix is equal to two (number of dimensions of the range), which equals the number of columns with pivot.

As you can see, Gaussian elimination is useful for finding if vectors are linearly independent, and not only that, but also for knowing WHICH vectors in the set can be kept and which ones discarded.

The null space are the solutions of the same system $\mathbf{Ax} = \mathbf{0}$. The columns without pivots (x_3 and x_4) can be given free parameters. The other variables can be obtained by inverse substitution:

$x_4 = \alpha$
$x_3 = \beta$
2nd row: $-3x_2 + x_3 - 3x_4 = 0 \to x_2 = \left(\frac{1}{3}\right)(x_3 - 3x_4) = -\alpha + \left(\frac{1}{3}\right)\beta$

1st row: $x_1 + x_2 - x_3 + 2x_4 = 0 \to x_1 = -x_2 + x_3 - 2x_4 = -\alpha + \left(\frac{4}{3}\right)\beta$

So the answer can be written in vector form:

$$\mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \alpha\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix} + \beta\begin{pmatrix}2/3\\1/3\\1\\0\end{pmatrix}$$

It is the equation of a plane in a 4-dimensional space.
Indeed, the theorem of dimensions is fulfilled, as rank($\mathbf{A}$) = 2 and nullity($\mathbf{A}$) = 2.

5) Example: Solve the following system of equations:

$$\begin{pmatrix}1 & 2 & -5 & -1 & 2\\0 & 1 & -2 & 1 & -4\\2 & -3 & 4 & 2 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}-3\\1\\9\end{pmatrix}$$

Write down the augmented matrix and perform Gaussian steps:

$$\left(\begin{array}{ccccc|c}1 & 2 & -5 & -1 & 2 & -3\\0 & 1 & -2 & 1 & -4 & 1\\2 & -3 & 4 & 2 & -1 & 9\end{array}\right) \xrightarrow{R_3-2R_1\to R_3} \left(\begin{array}{ccccc|c}1 & 2 & -5 & -1 & 2 & -3\\0 & 1 & -2 & 1 & -4 & 1\\0 & -7 & 14 & 4 & -5 & 15\end{array}\right)$$

$$\xrightarrow{R_3+7R_2\to R_3} \left(\begin{array}{ccccc|c}1 & 2 & -5 & -1 & 2 & -3\\0 & 1 & -2 & 1 & -4 & 1\\0 & 0 & 0 & 11 & -33 & 22\end{array}\right)$$

$$\xrightarrow{\left(\frac{1}{11}\right)R_3\to R_3} \left(\begin{array}{ccccc|c}1 & 2 & -5 & -1 & 2 & -3\\0 & 1 & -2 & 1 & -4 & 1\\0 & 0 & 0 & 1 & -3 & 2\end{array}\right)$$

The columns without pivots (x_3 and x_5) can be given free parameters. The other variables can be obtained by inverse substitution:

$x_5 = \alpha$
$x_3 = \beta$
3rd row: $x_4 - 3x_5 = 2 \ \rightarrow \ x_4 = 2 + 3x_5 = 2 + 3\alpha$
2nd row: $x_2 - 2x_3 + x_4 - 4x_5 = 1 \ \rightarrow \ x_2 = 2x_3 - x_4 + 4x_5 + 1 = -1 + 2\beta + \alpha$

1st row: $x_1 + 2x_2 - 5x_3 - x_4 + 2x_5 = -3 \rightarrow x_1 = -3 - 2x_2 + 5x_3 + x_4 + 2x_5 = 1 - \alpha + \beta$

So the answer can be written in vector form:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

It is the equation of a plane in a 5-dimensional space.
Note that the subspace spanned by the vectors multiplying α and β correspond to the null space of the transformation: as we know any solution can be added any multiple of a combination of those vectors.

Gaussian elimination can also be done with complex matrices.

6) **Problem: Solve** $\begin{pmatrix} i & -i & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} i \\ 3i \\ 0 \end{pmatrix}$

Solution:

We can write down the augmented matrix and perform Gaussian steps:

$$\left(\begin{array}{ccc|c} i & -i & 1 & i \\ 1 & -2 & 2 & 3i \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_2+iR_1\to R_2} \left(\begin{array}{ccc|c} i & -i & 1 & i \\ 0 & -1 & 2+i & -1+3i \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_3+iR_1\to R_3} \left(\begin{array}{ccc|c} i & -i & 1 & i \\ 0 & -1 & 2+i & -1+3i \\ 0 & 2 & 2+i & -1 \end{array}\right)$$

$$\xrightarrow{R_3+2R_2\to R_3} \left(\begin{array}{ccc|c} i & -i & 1 & i \\ 0 & -1 & 2+i & -1+3i \\ 0 & 0 & 6+3i & -3+6i \end{array}\right) \xrightarrow{R_3/3\to R_3} \left(\begin{array}{ccc|c} i & -i & 1 & i \\ 0 & -1 & 2+i & -1+3i \\ 0 & 0 & 2+i & -1+2i \end{array}\right)$$

Which we can solve by inverse substitution:

Third row: $x_3 = \frac{-1+2i}{2+i} = \frac{(-1+2i)(2-i)}{(2+i)(2-i)} = \frac{-2+i+4i+2}{4+1} = \frac{5i}{5} = i$

Second row: $-x_2 + (2+i)x_3 = (-1+3i) \rightarrow x_2 = (2+i)i - (-1+3i) = -1 + 2i + 1 - 3i = -i$

Third row: $ix_1 - ix_2 + x_3 = i \rightarrow ix_1 - 1 + i = i \rightarrow ix_1 = 1 \rightarrow x_1 = \frac{1}{i} = -i$

So, the solution is unique:

$$\mathbf{x} = \begin{pmatrix} -i \\ -i \\ i \end{pmatrix}$$

Gaussian elimination can be used to convert equations of lines/planes/etc. into parametric form:

7) Find the parametric equation of the following geometrical entity:

$$x - 4 = y = z + 1$$

From previous lectures we know that this is the equation of a line. However, we can apply the methods of this lecture to obtain that answer through another method:

The equation is in reality two different equations: $\begin{cases} x - 4 = y \\ y = z + 1 \end{cases}$

which can be written as: $\begin{cases} 1x - 1y + 0z = 4 \\ 0x + 1y - 1z = 1 \end{cases}$

Written as an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

It is already in row echelon form. We have a free parameter (the third column, with no pivot).

$z = \alpha$
2nd row: $y - z = 1 \;\rightarrow\; y = 1 + z = 1 + \alpha$
3rd row: $x - y = 4 \;\rightarrow\; x = 4 + y = 5 + \alpha$

So, the solution in vector form is, indeed as expected, the parametric equation of the line:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Interestingly, comparing the equation of the line $x - 4 = y = z + 1$ with the usual form $\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}$, we would have written the equation of the line as:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Both are, of course, the same line! Even though the specific equation looks different.

Remember that the specific choice of points $\mathbf{r}_0$ and the specific choice of vectors multiplying the free parameters, in this case $\mathbf{v}$, is not unique, because any point in the line is good for $\mathbf{r}_0$ and any vector in the line is good for $\mathbf{v}$.

8) A plane in 3D is given by the equation $z = 1 - y$. Find the parametric equation of this plane.

This is a very interesting question because it is unconventional. It is not written as a system of equations, however, the same algorithms we use to find the solutions can be applied to the case of matrices with one row!

The equation, written explicitly in terms of the x, y, z unknowns, is:

$$0x + 1y + 1z = 1$$

Written as an augmented matrix:

$$(0 \quad 1 \quad 1 \,|\, 1)$$

The second column has the pivot (the first non-zero number in the row, by definition). There are no more pivots because there are no more rows, so the other two columns correspond to free variables. So:

$x = \alpha$
$z = \beta$
1st row: $y + z = 1 \rightarrow y = 1 - z = 1 - \beta$

Therefore, written in vector form:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Which gives us two vectors that span the plane, and a point in the plane. Easy!

9) A plane in 3D is given by the equation $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0$. Find the parametric equation of this plane.

Again another unconventional question. But notice that this dot product is a linear equation:

$$2x + y + z = 0$$

Written as an augmented matrix:

$$(2 \quad 1 \quad 1 \,|\, 0)$$

The first column has a pivot, the other two columns correspond to free variables. So:

$z = \alpha$
$y = \beta$
1st row: $2x + y + z = 0 \rightarrow \; x = -\frac{1}{2}y - \frac{1}{2}z = -\frac{1}{2}\alpha - \frac{1}{2}\beta$

Therefore, written in vector form:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}$$

Which gives us two vectors that span the plane.

CALCULATION OF DETERMINANTS USING GAUSSIAN ELIMINATION

Gaussian elimination can be used for finding the determinant if we keep track of how each gaussian step affects the determinant of the resulting matrix:

- $R_i + \lambda R_j \rightarrow R_i$: The most common gaussian step **does not change the determinant**.
- $\lambda R_i \rightarrow R_i$: Scaling a row multiplies the determinant by the same scalar λ.
- $R_3 \leftrightarrow R_2$: Swapping two rows (or columns) multiplies the determinant by −1.
- **Once the matrix is triangular, the determinant is the product of the diagonal**.

10) Example: Calculate the determinant of $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ -100 & -100 & -150 \\ 4 & 6 & 8 \end{pmatrix}$ using Gaussian elimination.

$$\begin{vmatrix} 1 & 1 & 2 \\ -100 & -100 & -150 \\ 4 & 6 & 8 \end{vmatrix} \xrightarrow{\left(\frac{1}{50}\right) R_2 \rightarrow R_2} (50) \begin{vmatrix} 1 & 1 & 2 \\ -2 & -2 & -3 \\ 4 & 6 & 8 \end{vmatrix}$$

This gaussian step multiplied the determinant of the resulting matrix by $\left(\frac{1}{50}\right)$, so we multiply the determinant by 50 to keep it equal to the original determinant.

$$\xrightarrow{R_2 + 2R_1 \rightarrow R_2} (50) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 4 & 6 & 8 \end{vmatrix} \xrightarrow{R_3 - 4R_1 \rightarrow R_3} (50) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix}$$

These gaussian steps do not modify the determinant, so the determinants are equal. Finally we swap two rows, to make the matrix triangular, which changes the sign of the determinant. We multiply by (-1) to keep it equal.

$$\xrightarrow{R_3 \leftrightarrow R_2} (-1)(50) \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-1) \times 50 \times 2 = -100$$

Since the matrix is now triangular, the determinant is equal to the product of the diagonal elements.

For 3×3 matrices, this procedure seems more cumbersome than the usual cofactors method. However, for 4×4 and 5×5 this procedure is much easier. By the time you reach 20×20 this would be the only possible way of realistically calculating the determinant.

C. COMPUTATIONAL SOLVING OF SYSTEMS

Solving systems of equation by calculating the inverse (bruto force way using matrix of cofactors) involves calculating determinants, which are very expensive to compute.

If you estimate the number of operations required to solve a system of equations via a brute force calculation of the inverse, the answer is $\approx (n+1)!$ Where $n! = n(n-1)(n-2)\cdots(2)(1)$ stands for factorial.

On the other hand, solving a system by Gaussian elimination requires a number of operations $\approx n^3/3$.

Imagine having to solve system of equations with hundreds or thousands of variables, as often happens in a technical career:

Number of variables n	Gaussian elimination operations: $n^3/3$	Calculation of inverse using minors/cofactors : $(n+1)!$
10	333	$39\,916\,800$
100	$333\,333$	9.4×10^{159}
1000	$333\,333\,333$	4.0×10^{2570}

Even with powerful computers, we **MUST** use Gaussian elimination.

SOLVING LINEAR SYSTEM OF EQUATIONS IN PYTHON (GAUSSIAN ELIMINATION):

```
import numpy
A = np.array([[3,1], [1,2]])
b = np.array([9,8])
x = np.linalg.solve(A, b)
```

D. GAUSS-JORDAN ELIMINATION AND MATRIX INVERSE

Optional read: will not be in exam but can be very useful as a reference in your technical/scientific career.

Gaussian elimination can be further continued to turn a matrix in row echelon form into **reduced row echelon** form, in which **all pivots are always equal to 1** and have zeroes **both below and above**.

$$\left(\begin{array}{cccccccc|c} 1 & * & 0 & 0 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array}\right)$$

This is called Gauss-Jordan elimination. The solution to the system can then be read directly using each row to solve each variable (together with any free dependent variable).

If all columns have pivots (non-singular matrix), Gauss-Jordan elimination results in the identity matrix. The solution can then just be read from the right column of the independent coefficients:

$$(\mathbf{A}|\mathbf{b}) \xrightarrow{\text{Gauss-Jordan}} (\mathbf{I}|\mathbf{x})$$

In fact, if we start with an augmented matrix which uses the original matrix $\mathbf{A}$ placed next to the identity matrix $\mathbf{I}$, and then apply Gauss-Jordan elimination, we transform $\mathbf{A}$ into $\mathbf{I}$ and $\mathbf{I}$ into $\mathbf{A}^{-1}$.

$$(\mathbf{A}|\mathbf{I}) \xrightarrow{\text{Gauss-Jordan}} (\mathbf{I}|\mathbf{A}^{-1})$$

This is how computers compute the inverse of a matrix. Much faster than the usual recipe.

E. LU FACTORIZATION:

Optional read: will not be in exam but can be very useful as a reference in your technical/scientific career.

Each gaussian step of the form $R_a + \alpha R_b \to R_a$ can be codified into a matrix:

e.g. $R_2 + \alpha R_1 \to R_2$ can be written as: $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Therefore, the process of Gaussian elimination is the successive application of those steps, which can be seen as matrix multiplication of the different steps:

$$\begin{cases} R_2 + \alpha R_1 \to R_2 \\ R_3 + \beta R_1 \to R_3 \\ R_3 + \gamma R_2 \to R_3 \end{cases} \quad \text{can be written as:} \quad \mathbf{E} = \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the Gaussian elimination

$$(\mathbf{A}|\mathbf{b}) \xrightarrow{\text{Gaussian elimination}} (\mathbf{U}|\mathbf{c})$$

Can be represented as a matrix multiplication:

$$(\mathbf{A}|\mathbf{b}) \xrightarrow{\text{Gaussian elimination}} \mathbf{E}(\mathbf{A}|\mathbf{b}) = (\mathbf{EA} \mid \mathbf{Eb}) = (\mathbf{U}|\mathbf{c})$$

We have arrived at $\mathbf{EA} = \mathbf{U}$ so we can write $\mathbf{A} = \mathbf{E^{-1}U}$. It turns out that the matrix $\mathbf{E}$ has an inverse which is very easy to generate if we know the scaling factors of the Gaussian steps:

$$\begin{cases} R_2 + \alpha R_1 \to R_2 \\ R_3 + \beta R_1 \to R_3 \\ R_3 + \gamma R_2 \to R_3 \end{cases} \to \mathbf{E} = \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 \to \ \mathbf{E}^{-1} = (\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{E_1}^{-1}\mathbf{E_2}^{-1}\mathbf{E_3}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ -\beta & -\gamma & 1 \end{pmatrix}$$

With no need for actual calculation of inverses.

This is a lower triangular matrix which we call $\mathbf{L} = \mathbf{E^{-1}}$:

$$\begin{cases} R_2 + \alpha R_1 \to R_2 \\ R_3 + \beta R_1 \to R_3 \\ R_3 + \gamma R_2 \to R_3 \end{cases} \to \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ -\beta & -\gamma & 1 \end{pmatrix}$$

Such that Gaussian elimination $\mathbf{EA} = \mathbf{U}$ can be rewritten using $\mathbf{L} = \mathbf{E^{-1}}$ as:

$$\mathbf{A} = \mathbf{LU}$$

Note that $\mathbf{L}$ is a lower triangular matrix that can be constructed by the process of gaussian elimination (computationally efficient). The row echelon form $\mathbf{U}$ is an upper triangular matrix when the system has a unique solution (and thus all columns have pivots). The names of the matrices LU stand for “lower” and “upper” triangular.

Therefore, by doing Gaussian elimination, we can (with no need for extra operations) decompose the non-singular matrix $\mathbf{A}$ as a product of a lower triangular $\mathbf{L}$ and an upper triangular $\mathbf{U}$ matrix. This is called LU factorization. The advantage of calculating this factorization is that now we can use it to solve the system:

$$\mathbf{Ax} = \mathbf{b}$$
$$\mathbf{L}\,\underbrace{\mathbf{Ux}}_{\mathbf{y}} = \mathbf{b}$$

Which can be solved in two easy steps:

First solve $\mathbf{Ly} = \mathbf{b}$ to find $\mathbf{y}$. This is easy because $\mathbf{L}$ is triangular, so we can use inverse substitution.
Then solve $\mathbf{Ux} = \mathbf{y}$ to find $\mathbf{x}$. This is also easy because $\mathbf{U}$ is triangular, so we use inverse substitution.

This is very useful when you are going to solve the same system $\mathbf{Ax} = \mathbf{b}$ many times, perhaps millions of times, changing only the values of $\mathbf{b}$. The computation of $\mathbf{LU}$ is done together with gaussian elimination when solving the system for the first time and needs to be done only once. The subsequent solving of the system for any vector $\mathbf{b}$ requires only computationally fast inverse substitutions.

SWAPPING ROWS:

In the above steps we assumed that Gaussian elimination only involved steps of the type $R_a + \alpha R_b \to R_a$ with no substitution of rows. In practice we sometimes need to swap rows.

Fortunately, the Gaussian step of swapping rows also corresponds to a matrix:

e.g. $R_2 \leftrightarrow R_3$ can be written as: $\mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

And all necessary row permutations can be done at the beginning of the whole process via a simple permutation matrix $\mathbf{P}$, so that $\mathbf{PA}$ is ready for gaussian elimination with no further row permutations.

$$(\mathbf{A}|\mathbf{b}) \xrightarrow{\text{Permutations } \mathbf{P}} \mathbf{P}(\mathbf{A}|\mathbf{b}) \xrightarrow{\text{Gaussian elimination } \mathbf{E}} \mathbf{EP}(\mathbf{A}|\mathbf{b}) = (\mathbf{EPA} \mid \mathbf{EPb}) = (\mathbf{U}|\mathbf{c})$$

So that $\mathbf{EPA} = \mathbf{U}$. And remember that we called $\mathbf{E^{-1}} = \mathbf{L}$ so that the factorization becomes:

$$\mathbf{PA} = \mathbf{LU}$$

With $\mathbf{P}$ a permutation matrix, $\mathbf{L}$ a lower triangular matrix and $\mathbf{U}$ an upper triangular matrix.

EXAMPLE: COMPUTING PA=LU FACTORIZATION IN PYTHON:

```
import scipy
import scipy.linalg   # SciPy Linear Algebra Library

A = scipy.array([[7, 3, -1, 2],[3, 8, 1, -4],[-1, 1, 4, -1],[2, -4, -1, 6]])

P, L, U = scipy.linalg.lu(A) # returns matrices P, L and U
```

Now you are ready to solve the system $\mathbf{Ax} = \mathbf{b}$ as many times as you wish, requiring only inverse substitution.

2.5 EIGENVECTORS AND DIAGONALIZATION

A. CHANGE OF BASIS

Beautiful explanation of change of basis in 3blue1brown YouTube channel:
Change of basis | Essence of linear algebra, chapter 12 (13 min)

CHANGE BASIS OF VECTOR USING MATRIX MULTIPLICATION

A vector is an entity with magnitude and direction. Once we specify a basis, we can assign coordinates to the vector:

$$\mathbf{v} = (v_1, v_2)^T \text{ in basis } \{\mathbf{e}_1, \mathbf{e}_2\} \text{ represents the vector } \mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2.$$

Our usual basis is $\{\mathbf{e}_1, \mathbf{e}_2\} = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$.

But we can write the **SAME** vector, in **ANOTHER** basis $\{\mathbf{b}_1, \mathbf{b}_2\}$, giving two new coordinates $\mathbf{v}' = (v_1', v_2')$.

$$\mathbf{v}' = (v_1', v_2')^T \text{ in basis } \{\mathbf{b}_1, \mathbf{b}_2\} \text{ represents the vector } \mathbf{v} = v_1'\mathbf{b}_1 + v_2'\mathbf{b}_2.$$

Even though they look different in terms of coordinates, $\mathbf{v}$ and $\mathbf{v}'$ are the **SAME** vector. This is like writing the same concept in a different language.

How can we get the coordinates of the vector in the new language?

In previous problems, we solved a linear system of equations $\mathbf{v} = v_1'\mathbf{b}_1 + v_2'\mathbf{b}_2$ to find v_1', v_2'.

But there is an alternative (easier) way of doing it, using matrices. To figure it out it's best to start by thinking the other way around: How do you change a vector written in a different language $(v_1', v_2')^T$ in basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ into a vector written in our language $\{\mathbf{e}_1, \mathbf{e}_2\}$? We can multiply $(v_1', v_2')^T$ by a **change of basis matrix** $\mathbf{A}$, which is formed by placing the basis vectors we want to translate from as the columns of a matrix:

$$\underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{\substack{\text{vector written in} \\ \text{our basis} \\ v_1\mathbf{e}_1 + v_2\mathbf{e}_2}} = \underbrace{\begin{pmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{pmatrix}}_{\substack{\text{new basis vectors} \\ \mathbf{b}_1 \text{ and } \mathbf{b}_2 \\ \text{written in our basis}}} \cdot \underbrace{\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}}_{\substack{\text{vector written in} \\ \text{new basis} \\ v_1'\mathbf{b}_1 + v_2'\mathbf{b}_2}}$$

$$\mathbf{v} = \mathbf{A}\mathbf{v}'$$

$$\text{our basis} \leftarrow \text{new basis}$$

This is intuitive to understand if we realise that we literally want to find the vector $\mathbf{v} = v_1'\mathbf{b}_1 + v_2'\mathbf{b}_2$, which is exactly what this matrix multiplication is doing.

Then, to find the coordinates of a known vector in our language, translated to the new language, we just have to multiply by the inverse matrix!

$$\mathbf{v}' = \mathbf{A}^{-1}\mathbf{v}$$

1) **Example:** Expand the vector $\mathbf{v} = (-1,1,1)$ in the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ with $\mathbf{e_1} = (1,0,1)$, $\mathbf{e_2} = (0,1,1)$, and $\mathbf{e_3} = (1,1,0)$.

This is a problem we already did in the "N dimensional vectors" class, by solving a system of equations, solving for the linear coefficients a_1, a_2, a_3 such that:

$$\mathbf{v} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$$

Now we are going to solve it using change of basis matrices.

The change of basis matrix for $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\} \to \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ is given by placing $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ as columns (written in terms of the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ basis):

$$\mathbf{A} = \begin{pmatrix} | & | & | \\ \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The inverse of this matrix $\mathbf{A}^{-1}$ will convert $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\} \to \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$

Let's calculate the inverse. We start by computing the determinant using the first row:

$$\det \mathbf{A} = 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

The minors are the determinants obtained when crossing out the corresponding row and column:

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

The cofactors are the minors with alternating change of signs $(-1)^{i+j}$:

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

The inverse is equal to the transpose of the cofactor matrix, divided by the determinant:

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det \mathbf{A}} = \frac{\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}^T}{-2} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Therefore, we can expand vector ***v*** into the new basis by applying the change $\mathbf{A}$:

$$\mathbf{v}' = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \mathbf{A^{-1}v} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

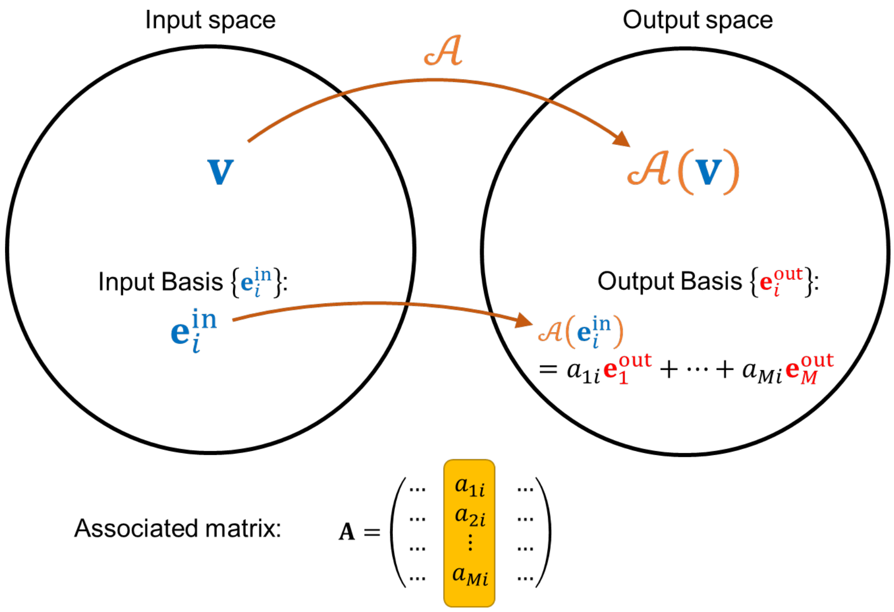
The great thing about having the matrix $\mathbf{A^{-1}}$ is that we can immediately translate any other vector $\mathbf{v}$ doing a simple matrix-vector multiplication.

It is curious to note that this time, $\mathbf{A}$ is representing a **change of basis** which leaves the actual vector entity unchanged, instead of representing a linear transformation of vectors. However, it can also be interpreted as a transformation.

CHANGING BASIS OF A MATRIX

One same vector can be written in different languages/basis. Similarly, one **same linear transformation** can be written in different bases too, resulting in different matrix coefficients.

In other words, a linear transformation can be written as a matrix ONLY AFTER we define a basis for the input and output spaces. The exact coefficients in this matrix will depend on the basis chosen for **BOTH** the input and the output vector spaces.



Therefore, the coefficients of a given matrix represent a given linear transformation ONLY for a given input and output basis. The examples we have seen so far always assumed the usual $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ basis for both input and output.

In many cases, an appropriate choice of basis means that the matrix associated with a linear transformation can be written in a simpler way.

Let's focus on square matrices in which the input and output spaces are the same dimension and are represented using the same basis vectors: e.g. 2D to 2D transformations typically written in $\{\mathbf{x}, \mathbf{y}\}$ basis, both input and output.

The question is: if we know $\mathbf{M}$ in our basis, how can we translate it to the new language $\mathbf{M}'$?

Transformation $\mathbf{w} = \mathbf{M}\mathbf{v}$ where vectors and matrices are written in "our" basis $\{\mathbf{e}_1, \mathbf{e}_2\}$

Transformation $\mathbf{w}' = \mathbf{M}'\mathbf{v}'$ where vectors and matrices are written in "new" basis $\{\mathbf{b}_1, \mathbf{b}_2\}$

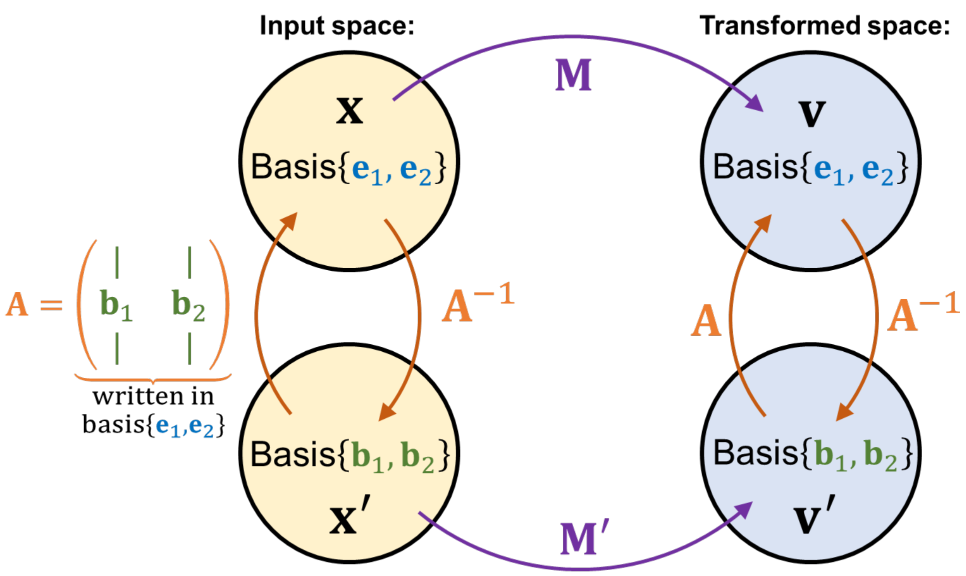
We can figure it out by using the change of basis matrix $\mathbf{A}$ (new language $\rightarrow$ our language):

$$\overbrace{\mathbf{w}'}^{\text{transformed vector in new basis}} = \underbrace{\mathbf{A}^{-1}\underbrace{\mathbf{M}\underbrace{\mathbf{A}\overbrace{\mathbf{v}'}^{\text{original vector in new basis}}}_{\text{original vector in our basis}}}_{\text{transformed vector in our basis}}}_{\text{transformed vector in new basis}}$$

<u>SIMILARITY TRANSFORMATIONS</u>

$$\mathbf{M}' = \mathbf{A}^{-1}\mathbf{M}\mathbf{A}$$

$$\mathbf{M} = \mathbf{A}\mathbf{M}'\mathbf{A}^{-1}$$



The matrices $\mathbf{M}$ and $\mathbf{M}'$ represent the **<u>SAME linear transformation</u>**, just in different basis!

$\mathbf{M}$ and $\mathbf{M}'$ are called **<u>SIMILAR MATRICES</u>**.

Since they represent the same linear transformation, they share all properties of the transformation which are independent of the basis:

Similar matrices always have the same…

- Range and null-space (but written in their corresponding basis)
- Rank and nullity
- Determinant (the "amount of stretching/squashing" of the transformation)
- Trace
- Eigenvalues
- Eigenvectors (but written in the corresponding basis)

ORTHOGONAL CHANGE-OF-BASIS MATRIX

When the new basis $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_N\}$ is orthonormal, the change-of-basis matrix

$$\mathbf{A} = \begin{pmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_N \\ | & | & & | \end{pmatrix}$$

will have orthonormal vectors as its columns. Therefore, it is a **unitary matrix**, and as we know $\mathbf{A}^{-1} = \mathbf{A}^{\dagger}$, so the similarity transformation becomes $\mathbf{M} = \mathbf{A}\mathbf{M}'\mathbf{A}^{\dagger}$ and is even easier to calculate.

Note: when considering real matrices, **unitary matrices** are called **orthogonal matrix**, and the property is $\mathbf{A}^{-1} = \mathbf{A}^{T}$.

2) **Example: Change the basis of the input and output space of matrix $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$** using a new basis $\{\mathbf{b}_1 = (2,1), \mathbf{b}_2 = (1,2)\}$. Check the determinant and trace stay the same.

Let's build the matrices which change the basis $\{\mathbf{b}_1, \mathbf{b}_2\} \leftrightarrow \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$.

$$\mathbf{A} = \begin{pmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ translates basis } \{\mathbf{b}_1, \mathbf{b}_2\} \to \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$$

$$\mathbf{A}^{-1} = \tfrac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \text{ calculated by doing the inverse, translates basis } \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\} \to \{\mathbf{b}_1, \mathbf{b}_2\}$$

Now let's convert our linear transformation $\mathbf{M}$ into the language of basis $\{\mathbf{b}_1, \mathbf{b}_2\}$

$$\mathbf{M}' = \mathbf{A}^{-1}\mathbf{M}\mathbf{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

Indeed, $\det(\mathbf{M}) = \det(\mathbf{M}') = 3$ and $\mathrm{trace}(\mathbf{M}) = \mathrm{trace}(\mathbf{M}') = 3$.

3) **Orthonormal basis example: Change the basis of the input and output space of matrix $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$** using a new basis $\left\{\mathbf{b}_1 = \frac{1}{\sqrt{2}}(1,1), \mathbf{b}_2 = \frac{1}{\sqrt{2}}(-1,1)\right\}$. Check the determinant and trace stay the same.

Let's build the matrices which change the basis $\{\mathbf{b}_1, \mathbf{b}_2\} \leftrightarrow \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$.

$$\mathbf{A} = \begin{pmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{pmatrix} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ translates basis } \{\mathbf{b}_1, \mathbf{b}_2\} \to \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$$

Because $\{\mathbf{b}_1, \mathbf{b}_2\}$ are orthonormal, the real matrix $\mathbf{A}$ is orthogonal, and so:

$$\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ translates basis } \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\} \to \{\mathbf{b}_1, \mathbf{b}_2\}$$

Now let's convert our linear transformation $\mathbf{M}$ into the language of basis $\{\mathbf{b}_1, \mathbf{b}_2\}$

$$\mathbf{M}' = \mathbf{A}^{-1}\mathbf{M}\mathbf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & -3 \\ 1 & 3 \end{pmatrix}$$

Indeed, $\det(\mathbf{M}) = \det(\mathbf{M}') = 3$ and $\mathrm{trace}(\mathbf{M}) = \mathrm{trace}(\mathbf{M}') = 3$.

SIMILARITY TRANSFORMATIONS FOR RECTANGULAR MATRICES

Let's generalize this to rectangular matrices. Assume a general $M \times N$ matrix $\mathbf{M}$ represents a linear transformation which acts on an N dimensional input space written using basis $\{\mathbf{e}_i^{\text{in}}\}$ and its output is in an M dimensional vector space written using basis $\{\mathbf{e}_i^{\text{out}}\}$. In previous examples we used $\{\mathbf{e}_i^{\text{in}}\} = \{\mathbf{e}_i^{\text{out}}\}$, but in general they could be different. Find out how to write the same linear transformation using a matrix working with an input space basis $\{\mathbf{b}_i^{\text{in}}\}$ and an output space basis $\{\mathbf{b}_i^{\text{out}}\}$.

First, we compute the translation matrix for the input and output space:

$$\mathbf{A}^{\text{in}}_{N\times N} = \underbrace{\begin{pmatrix} | & & | \\ \mathbf{b}_1^{\text{in}} & \dots & \mathbf{b}_N^{\text{in}} \\ | & & | \end{pmatrix}}_{\substack{\text{new basis vectors } \mathbf{b}_i^{\text{in}} \\ \text{written in basis } \mathbf{e}_i^{\text{in}}}} \text{ : translates from new language } \{\mathbf{b}_i^{\text{in}}\} \text{ to our language } \{\mathbf{e}_i^{\text{in}}\}.$$

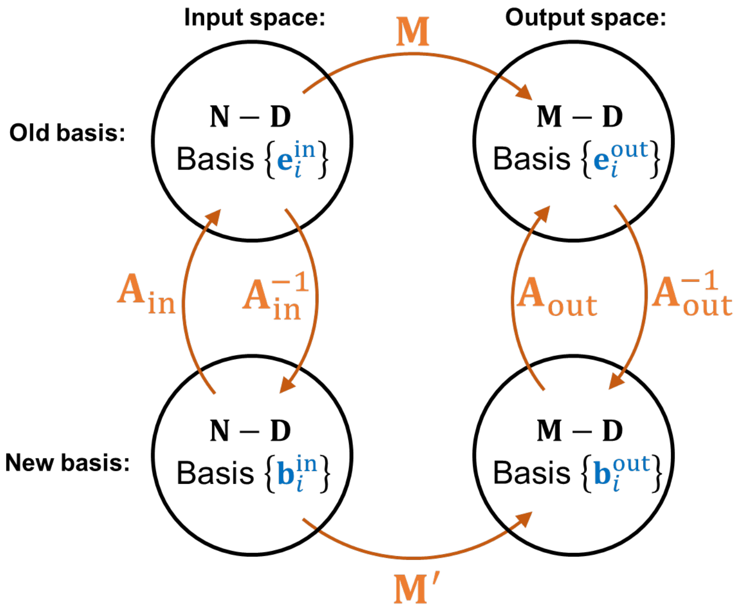
$$\mathbf{A}^{\text{out}}_{M\times M} = \underbrace{\begin{pmatrix} | & & | \\ \mathbf{b}_1^{\text{out}} & \dots & \mathbf{b}_M^{\text{out}} \\ | & & | \end{pmatrix}}_{\substack{\text{new basis vectors } \mathbf{b}_i^{\text{in}} \\ \text{written in basis } \mathbf{e}_i^{\text{in}}}} \text{ : translates from new language } \{\mathbf{b}_i^{\text{out}}\} \text{ to our language } \{\mathbf{e}_i^{\text{out}}\}.$$

If we want to translate in the other direction, we need to use the inverse matrices.

Then, to find the transformation matrix in the new language, we can do a chain of operations: (i) translation in input space from the new language to our language, (ii) application of transformation in our language, and (iii) translation of the output back to new language. Each operation is added on the left as a pre-multiplication, so we get:

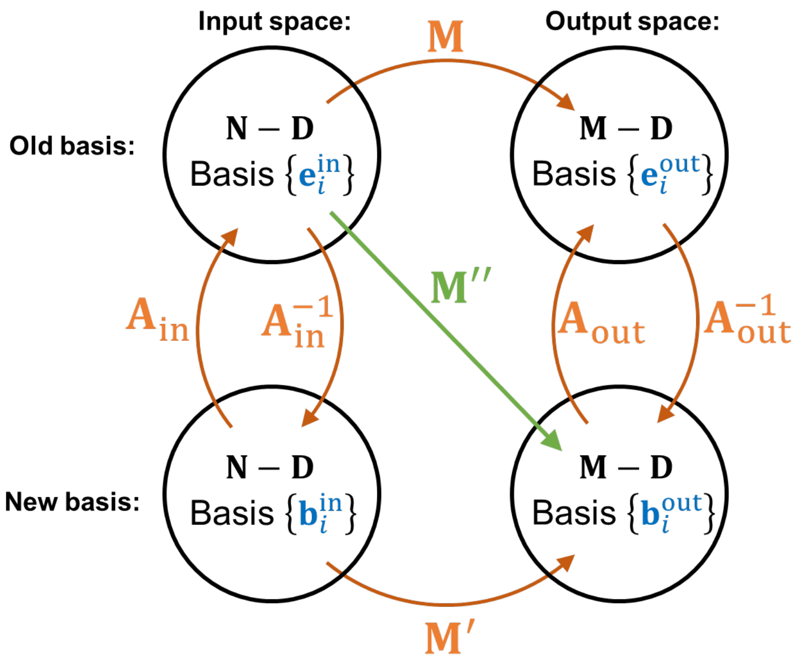
$$\mathbf{M}' = (\mathbf{A}^{\text{out}})^{-1}\mathbf{M}\mathbf{A}^{\text{in}}$$

You can see that all the sizes in the multiplication are correct.



Example to deepen our understanding: Let's mix things up! Find the matrix $\mathbf{M}''$ (representing the same linear transformation as $\mathbf{M}$) which is defined using our language for the input space basis $\{\mathbf{e}_i^{\mathrm{in}}\}$ **but** gives the result in the output space using the new language $\{\mathbf{b}_i^{\mathrm{out}}\}$. This new matrix combines the transformation with a change of basis. Write this new matrix in terms of $\mathbf{M}$ and also in terms of $\mathbf{M}'$. Once you find both expressions, check that they are equal using the relation between $\mathbf{M}$ and $\mathbf{M}'$.

Things seem to be getting complicated. But everything becomes crystal clear if we do a diagram of the different vector spaces and their transformations, and find paths from desired input to desired output, where each additional step is a matrix multiplication on the left:



We need to find an expression for the path $\mathbf{M}''$ which starts top left and ends in bottom right.

To write $\mathbf{M}''$ in terms of $\mathbf{M}$, we go right and then down:

$$\mathbf{M}'' = (\mathbf{A}^{\mathrm{out}})^{-1}\mathbf{M}$$

To write $\mathbf{M}''$ in terms of $\mathbf{M}'$, we go down and then right:

$$\mathbf{M}'' = \mathbf{M}'\left(\mathbf{A}^{\mathrm{in}}\right)^{-1}$$

Check that both are equal via the relation $\mathbf{M}' = (\mathbf{A}^{\mathrm{out}})^{-1}\mathbf{M}\mathbf{A}^{\mathrm{in}}$:

$$\mathbf{M}'' = \mathbf{M}'\left(\mathbf{A}^{\mathrm{in}}\right)^{-1} = (\mathbf{A}^{\mathrm{out}})^{-1}\mathbf{M}\,\underbrace{\mathbf{A}^{\mathrm{in}}\left(\mathbf{A}^{\mathrm{in}}\right)^{-1}}_{\mathbf{I}} = (\mathbf{A}^{\mathrm{out}})^{-1}\mathbf{M}$$

B. EIGENVECTORS INTUITIVE UNDERSTANDING: $\mathbf{Av} = \lambda\mathbf{v}$

In general, a linear transformation modifies an input vector such that both magnitude and direction are modified. But sometimes, a linear transformation $\mathcal{A}$ has the property that, for some special vectors in the input space, **it does not change their direction, only their magnitude**. For these special vectors (**called eigenvectors**) the transformation $\mathcal{A}(\mathbf{v}) \equiv \mathbf{Av}$ is equivalent to a simple scaling $\lambda\mathbf{v}$, with a specific scaling coefficient λ (**called eigenvalue**). This is summarized as $\mathbf{Av} = \lambda\mathbf{v}$**.**

Knowing what these vectors are can give us a lot of information about the transformation.

4) Example: ISOTROPIC SCALING OF SPACE – determine eigenvectors and eigenvalues by intuition

Consider a transformation which scales the whole space by a factor K.

$$\mathbf{A} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$

Every vector in the input space undergoes a change in amplitude with no change in direction.

Therefore, every vector in space fulfils:

$$\mathbf{Av} = K\mathbf{v}.$$

In this case, every vector in space $\mathbf{v} \in \text{span}\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ is an eigenvector, with eigenvalue K. When specifying the eigenvectors, we simply list the vectors that span the subspace:

Eigenvectors	Eigenvalue
$\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$	K

5) Example: ROTATION IN 2D – determine eigenvectors and eigenvalues by intuition

Consider the transformation which rotates the whole 2D space:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This transformation modifies the direction of every possible input vector (except the trivial case $\mathbf{v} = \mathbf{0}$) and therefore it has no eigenvectors (at least, no real ones… see later).

6) Example: ROTATION IN 3D – determine eigenvectors and eigenvalues by intuition

Consider the transformation which rotates 3D space around the z-axis:

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Any vector parallel to the z-axis is unchanged, and fulfils:

$$\mathbf{Av} = \mathbf{v}$$

Therefore, the vectors parallel to the z-axis are an eigenvector with eigenvalue $\lambda = 1$:

Eigenvector	Eigenvalue
$\hat{\mathbf{z}}$	1

In fact, in general, any rotation around an axis $\mathbf{n}$ will not modify the direction of vectors in the direction of the axis $\mathbf{n}$, so that $\mathbf{An} = \mathbf{n}.$ So, any vector in $\text{span}\{\mathbf{n}\}$ is an eigenvector with eigenvalue $\lambda = 1$.

Eigenvector	Eigenvalue
$\mathbf{n}$	1

7) Example: ANISOTROPIC SCALING (easy case) – determine eigenvectors and eigenvalues by intuition

Consider a transformation which stretches space by a factor of 3 along x, and by a factor $\frac{1}{2}$ along y.

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Any vector parallel to x will fulfil:

$$\mathbf{Av} = 3\mathbf{v}.$$

Any vector parallel to y will fulfil:

$$\mathbf{Av} = \left(\frac{1}{2}\right)\mathbf{v}.$$

Therefore:

Eigenvectors	Eigenvalues
$\hat{\mathbf{x}}$	3
$\hat{\mathbf{y}}$	1/2

8) Example: PROJECTION TO A LINE (easy case) – determine eigenvectors and eigenvalues by intuition

Consider the transformation which projects the entire 2D space into the x-axis:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

It is obvious that this transformation changes the direction of every vector except those that point along the x axis for which $\mathbf{Av} = \mathbf{v}$ and, less obviously, those which point along the y axis, which are squashed to zero and also fulfil $\mathbf{Av} = \mathbf{0v} = \mathbf{0}$. Therefore, this transformation has two eigenvectors:

Eigenvectors	Eigenvalue
$\hat{\mathbf{x}}$	1
$\hat{\mathbf{y}}$	0

9) Example: SKEW – determine eigenvectors and eigenvalues by intuition

Consider the skew transformation:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

This transformation skews vectors to the right. It however does not change vectors pointing purely along the x axis. In fact $\mathbf{A}\hat{\mathbf{x}} = 1\hat{\mathbf{x}}$. Therefore:

Eigenvectors	Eigenvalue
$\hat{\mathbf{x}}$	1

NOTICE COMMON PATTERNS IN ALL PREVIOUS EXAMPLES (and always true in general):

- **The eigenvectors are telling us valuable information** about the transformation: the axis of rotation, the axis of projection, the axis of scaling, the direction parallel and normal to a mirror reflection, the direction of skew, …
- **The eigenvectors associated with each eigenvalue span an entire subspace (called an eigenspace) of the input space. Any vector in this subspace is an eigenvector.**
    - This is due to linearity of the transformation: if $\mathbf{v}$ is an eigenvector $\mathbf{Av} = \lambda\mathbf{v}$, then any scaled version of $\mathbf{v}$ will be an eigenvector too: $\mathbf{A}(\alpha\mathbf{v}) = \lambda(\alpha\mathbf{v})$.
    - This means that the system of equations $\mathbf{Av} = \lambda\mathbf{v}$ always has either a single trivial solution $\mathbf{v} = 0$ when there are no eigenvectors, or it has infinite solutions, whole subspaces (infinite line, plane, etc.).
- **The eigenvectors whose eigenvalue is 0 are the null space of the linear transformation**, as they are solutions to $\mathbf{Av} = 0\mathbf{v} = \mathbf{0}$.
    - If an eigenvalue equal to 0 exists, the matrix is singular.
- $N \times N$ matrices have at most N linearly independent eigenvectors, others have less than N down to at least 1.
- **Whenever a column of the matrix has zeroes everywhere except on the diagonal term, the associated basis vector for that column is an eigenvector, whose eigenvalue is the number at the diagonal.** This is trivial if you think about the columns of the matrix as $\mathcal{A}(\hat{\mathbf{e}}_i)$.

Example:

$$\mathbf{A} = \begin{pmatrix} | & | & | & | \\ \mathcal{A}(\mathbf{e}_1) & \mathcal{A}(\mathbf{e}_2) & \mathcal{A}(\mathbf{e}_3) & \mathcal{A}(\mathbf{e}_4) \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} * & 0 & * & 0 \\ * & a & * & 0 \\ * & 0 & * & 0 \\ * & 0 & * & b \end{pmatrix}$$

$\mathcal{A}(\mathbf{e}_2) = 0\mathbf{e}_1 + a\mathbf{e}_2 + 0\mathbf{e}_3 + 0\mathbf{e}_4 = a\mathbf{e}_2$, therefore $\mathbf{e}_2$ is an eigenvector with eigenvalue a.

$\mathcal{A}(\mathbf{e}_4) = 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3 + b\mathbf{e}_4 = b\mathbf{e}_4$ therefore $\mathbf{e}_4$ is an eigenvector with eigenvalue b.

C. CALCULATION OF EIGENVECTORS AND EIGENVALUES

Eigenvectors, as we have seen, are vectors such that $\mathbf{Av} = \lambda\mathbf{v}$. We need to solve that equation for any possible value of λ and $\mathbf{v}$ (evidently $\mathbf{A}$ must be a square matrix):

$$\mathbf{Av} = \lambda\mathbf{v}$$

We want to write it as a system of equations, (matrix)*(unknowns) = (independent coefficients)

$$\mathbf{Av} - \lambda\mathbf{v} = \mathbf{0}$$

$$\mathbf{Av} - \lambda\mathbf{Iv} = \mathbf{0}$$

$\mathbf{I}$ being the identity matrix, and the matrix transformation $\lambda\mathbf{I}$ is identical to scaling the vector by λ.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

FIRST: FIND THE EIGENVALUES

Now we have it in the desired form. The matrix $(\mathbf{A} - \lambda\mathbf{I})$ acts as coefficients of the linear system of equations. This matrix can be seen as a new transformation. It looks like this:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} - \lambda \end{pmatrix}$$

Earlier we proved that either there is a single trivial solution $\mathbf{v} = \mathbf{0}$, which will happen when $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$, or there are infinite solutions (a whole subspace) when $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. We are interested on the second case.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$c_N\lambda^N + c_{N-1}\lambda^{N-1} + \cdots + c_1\lambda + c_0 = 0$ [N-th degree polynomial]

$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N) = 0$ [Factorised into N complex roots (some can be repeated)]

This is called the **characteristic polynomial** $p(\lambda)$ of the matrix $\mathbf{A}$.

The roots $\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ are the N eigenvalues (some can be repeated, called degenerate)

Once we have found the N eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$, we need to find the eigenvectors associated to each of them.

SECOND: FIND THE EIGENVECTORS FOR EACH EIGENVALUE

For each eigenvalue λ_i that we found, we need to solve the linear systems $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}$ to find the corresponding eigenvectors. For example, we can use Gaussian elimination.

We know that the system must have infinite solutions, because the determinant is zero: the solution will be a subspace, i.e. a span of at least one vector. This vector (or vectors) will be eigenvectors $\mathbf{v}_i$ associated to the eigenvalue λ_i.

Note that what is fixed for a given eigenvalue is the subspace, which is the solution to $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}$ (called the eigenspace). We have some freedom in choosing the vectors which span it. For example, we can always scale an eigenvector and it will still be an eigenvector of the same eigenvalue.

10) ISOTROPIC SCALING OF SPACE – determine eigenvectors and eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}.$$

First, find the eigenvalues by solving the characteristic polynomial:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} K-\lambda & 0 \\ 0 & K-\lambda \end{vmatrix} = (K-\lambda)(K-\lambda) = 0$$

It has two degenerate roots, therefore two degenerate eigenvalues $\lambda_1 = \lambda_2 = K$. Now to find the eigenvectors, we need to find the span of solutions to the system:

$$(\mathbf{A} - K\mathbf{I})\mathbf{v} = \mathbf{0} \;\rightarrow\; \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

No need to do gaussian elimination! Two columns, both without pivots. The solution has two degrees of freedom $v_1 = \alpha$ and $v_2 = \beta$ so that $\mathbf{v} = \alpha(1,0)^T + \beta(0,1)^T$ which are the two eigenvectors.

Eigenvectors	**Eigenvalue**
$\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$	K

11) ANISOTROPIC SCALING (oriented along the axes) – determine eigenvectors and eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}$

First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & \frac{1}{2}-\lambda \end{vmatrix} = (3-\lambda)\left(\frac{1}{2}-\lambda\right)$$

So the two eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = \frac{1}{2}$.

For eigenvalue $\lambda_1 = 3$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Clearly the first column has no pivot and is a free variable, so $x_1 = \alpha$ and $x_2 = 0$.

$$\mathbf{x} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{span}\{\mathbf{v}_1\}$$

For eigenvalue $\lambda_2 = \frac{1}{2}$:

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = \begin{pmatrix} 5/2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Clearly the second column has no pivot and is a free variable, so $x_2 = \alpha$ and $x_1 = 0$.

$$\mathbf{x} = \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{span}\{\mathbf{v}_2\}$$

Eigenvector	**Eigenvalue**
$(1,0)^T$	3
$(0,1)^T$	$1/2$

12) PROJECTION TO A LINE (easy case) – determine eigenvectors and eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 0-\lambda \end{vmatrix} = (1-\lambda)(0-\lambda)$$

So the two eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$.

For eigenvalue $\lambda_1 = 1$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Clearly the first column has no pivot and is a free variable, so $x_1 = \alpha$ and $x_2 = 0$.

$$\mathbf{x} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{span}\{\mathbf{v}_1\}$$

For eigenvalue $\lambda_2 = 0$:

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Clearly the second column has no pivot and is a free variable, so $x_2 = \alpha$ and $x_1 = 0$.

$$\mathbf{x} = \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{span}\{\mathbf{v}_2\}$$

Eigenvector	Eigenvalue
$(1,0)^T$	1
$(0,1)^T$	0

13) SKEW – determine eigenvectors and eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)$$

So there is a double degenerate eigenvalue $\lambda_1 = \lambda_2 = 1$. The associated eigenvector can be obtained by solving:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Clearly the first column has no pivot and is a free variable, so $x_1 = \alpha$ and $x_2 = 0$.

$$\mathbf{x} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{span}\{\mathbf{v}_1\}$$

Eigenvector	Eigenvalue
$(1,0)^T$	1

14) **MIRROR REFLECTION WITH RESPECT TO $x = y$ LINE** – determine eigenvectors and eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = (0-\lambda)(0-\lambda) - 1 = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

So the two eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.

For eigenvalue $\lambda_1 = 1$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{R_2+R_1\to R_2} \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So the second column has no pivot and is a free variable, $x_2 = \alpha$, and from the first row $x_1 = \alpha$.

$$\mathbf{x} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathrm{span}\{\mathbf{v}_1\}$$

For eigenvalue $\lambda_2 = -1$:

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2-R_1\to R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So the second column has no pivot and is a free variable, $x_2 = \alpha$ and from the first row $x_1 = -\alpha$.

$$\mathbf{x} = \alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \mathrm{span}\{\mathbf{v}_2\}$$

Eigenvector	**Eigenvalue**
$(1,1)^T$	1
$(-1,1)^T$	-1

As one would intuitively expect for this mirror reflection. It tells us the direction parallel (eigenvalue 1) and normal (eigenvalue -1) to the mirror! Because vectors parallel to the mirror are unchanged, while vectors normal to the mirror are flipped.

15) **PROJECTION INTO THE LINE $y = mx$ –** determine eigenvectors and eigenvalues of the matrix**:**

$$\mathbf{A}(m) = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$$

First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} \dfrac{1}{1+m^2} - \lambda & \dfrac{m}{1+m^2} \\ \dfrac{m}{1+m^2} & \dfrac{m^2}{1+m^2} - \lambda \end{vmatrix} = \left(\frac{1}{1+m^2} - \lambda\right)\left(\frac{m^2}{1+m^2} - \lambda\right) - \left(\frac{m}{1+m^2}\right)^2 = 0$$

$$\left(\frac{m}{1+m^2}\right)^2 - \lambda\frac{1}{1+m^2} - \lambda\frac{m^2}{1+m^2} + \lambda^2 - \left(\frac{m}{1+m^2}\right)^2 = 0$$
$$\lambda\frac{-1-m^2}{1+m^2} + \lambda^2 = 0$$
$$\lambda^2 - \lambda = 0 \quad \rightarrow \quad \lambda(\lambda-1) = 0$$

So the two eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$. (**Shortcut**: use trace and det of matrix to deduce that $\det(\mathbf{A}) = \lambda_1\lambda_2 = 0$ and $\mathrm{trace}(\mathbf{A}) = \lambda_1 + \lambda_2 = 1$; properties taught later)

For eigenvalue $\lambda_1 = 1$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \begin{pmatrix} \frac{1}{1+m^2} - 1 & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} - 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

$$\left(\begin{array}{cc|c} \frac{-m^2}{1+m^2} & \frac{m}{1+m^2} & 0 \\ \frac{m}{1+m^2} & \frac{-1}{1+m^2} & 0 \end{array}\right) \xrightarrow{R_2 + \left(\frac{1}{m}\right)R_1 \to R_2} \left(\begin{array}{cc|c} \frac{-m^2}{1+m^2} & \frac{m}{1+m^2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So the second column has no pivot and is a free variable, $x_2 = \alpha$, and

From the first row: $-m^2x_1 + m\alpha = 0 \rightarrow \; -mx_1 + \alpha = 0 \; \rightarrow \; x_1 = \alpha/m$

$$\mathbf{x} = \alpha\begin{pmatrix} 1/m \\ 1 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 1/m \\ 1 \end{pmatrix}\right\} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ m \end{pmatrix}\right\} = \mathrm{span}\{\mathbf{v}_1\} \text{ is our first eigenvector}$$

Notice that we are free to scale the eigenvectors to simplify the expression if we wish, as we did above.

For eigenvalue $\lambda_2 = 0$:

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

$$\left(\begin{array}{cc|c} \frac{1}{1+m^2} & \frac{m}{1+m^2} & 0 \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} & 0 \end{array}\right) \xrightarrow{R_2 - mR_1 \to R_2} \left(\begin{array}{cc|c} \frac{1}{1+m^2} & \frac{m}{1+m^2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So the second column has no pivot and is a free variable, $x_2 = \alpha$ and

From the first row: $x_1 + m\alpha = 0 \rightarrow \; x_1 = -m\alpha$

$$\mathbf{x} = \alpha\begin{pmatrix} -m \\ 1 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} -m \\ 1 \end{pmatrix}\right\} = \mathrm{span}\{\mathbf{v}_2\} \text{ is our second eigenvector}$$

Eigenvectors	Eigenvalue
$(1, m)^{\mathrm{T}}$	1
$(-m, 1)^{\mathrm{T}}$	0

Notice that, by finding the eigenvectors, we have found the direction of the line of projection (with eigenvalue 1 because vectors along that line are not changed) and the direction normal to the line of projection (with eigenvalue 0 because vectors in that direction are projected into the origin).

Sometimes, the eigenvalues and eigenvectors of a real matrix can be complex (this usually happens with rotations).

As a general rule for real matrices, **when two eigenvalues are complex conjugates of one another, then the associated eigenvectors are also complex conjugates of one another**.

16) 2D ROTATION: Calculate the eigenvectors and eigenvalues for the matrix:

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Solution: First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = 1 - 2\lambda\cos\theta + \lambda^2$$

Quadratic polynomial with solutions:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2} = e^{\pm i\theta}$$

So that $\det(\mathbf{A} - \lambda\mathbf{I}) = \left(e^{i\theta} - \lambda\right)\left(e^{-i\theta} - \lambda\right) = 0$. The eigenvalues are the two roots: $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{-i\theta}$. The associated eigenvectors can then be obtained for each:

For eigenvalue $\lambda_1 = e^{i\theta}$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x_1} = \mathbf{0}$$

$$\begin{pmatrix} \cos\theta - e^{i\theta} & -\sin\theta \\ \sin\theta & \cos\theta - e^{i\theta} \end{pmatrix}\mathbf{x_1} = \mathbf{0}$$

We don't really need to go through Gauss elimination, because we know that the determinant is zero (as we imposed that as a condition for finding λ_1) so we know we have one degree of freedom which we can use straight away:

$$x_{12} = \alpha$$
$$x_{11}\left(\cos\theta - e^{i\theta}\right) - x_{12}\sin\theta = 0$$

Therefore: $x_{11} = \alpha\frac{\sin\theta}{\cos\theta - e^{i\theta}} = \alpha\frac{\frac{e^{i\theta} - e^{-i\theta}}{2i}}{\frac{e^{i\theta} + e^{-i\theta}}{2} - e^{i\theta}} = -i\alpha\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta} - 2e^{i\theta}} = -i\alpha\frac{e^{i\theta} - e^{-i\theta}}{-e^{i\theta} + e^{-i\theta}} = i\alpha.$

Written in vector form:

$$\mathbf{x} = \alpha\begin{pmatrix} i \\ 1 \end{pmatrix} = \mathrm{span}\{\mathbf{v}_1\}$$

For eigenvalue $\lambda_2 = e^{-i\theta}$ we find, after an almost identical procedure:

$$\mathbf{x} = \alpha\begin{pmatrix} -i \\ 1 \end{pmatrix} = \mathrm{span}\{\mathbf{v}_2\}$$

Therefore, the eigenvalues and eigenvectors are:

Eigenvector	Eigenvalue
$(i, 1)^T$	$e^{i\theta}$
$(-i, 1)^T$	$e^{-i\theta}$

How is it possible that a rotation has eigenvectors? They are complex!

We thought that rotation had no eigenvectors, but that is because our intuition was limited to real space. It turns out that rotations DO have complex vectors whose direction is left unchanged!

Let's check with an example. The rotation $\theta = \pi/2$ (90 degree rotation counter-clockwise).

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The two eigenvalues and eigenvectors are:

$$\lambda_1 = i \quad \rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\lambda_2 = -i \quad \rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Indeed, we can check the result:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \quad \rightarrow \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \quad \rightarrow \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix}$$

17) ROTATION IN 3D – determine eigenvectors and eigenvalues of the matrix:

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)[(\cos\theta - \lambda)^2 + \sin^2\theta] = (1-\lambda)[1 - 2\lambda\cos\theta + \lambda^2]$$

The quadratic polynomial between square brackets has the solutions:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2} = e^{\pm i\theta}$$

Therefore, the characteristic polynomial is:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \left(e^{i\theta} - \lambda\right)\left(e^{-i\theta} - \lambda\right)(1-\lambda) = 0$$

So, the eigenvalues are $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$ and $\lambda_3 = 1$

And the associated eigenvectors:

For eigenvalue $\lambda_1 = e^{i\theta}$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} \cos\theta - e^{i\theta} & -\sin\theta & 0 \\ \sin\theta & \cos\theta - e^{i\theta} & 0 \\ 0 & 0 & 1 - e^{i\theta} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

We don't really need to go through Gauss elimination, because we know that the determinant is zero (as we imposed that as a condition for finding λ_1) so we know we have one degree of freedom which we can use straight away. Which column? We know that the third column has a pivot. Therefore the first two columns must be linearly dependent and one will not have a pivot. (If you don't trust this reasoning, go ahead and do gaussian elimination). The solution is:

$$x_2 = \alpha$$

$$x_3 = 0$$
$$x_1\left(\cos\theta - e^{i\theta}\right) - x_2 \sin\theta = 0$$

Therefore: $x_1 = \alpha\frac{\sin\theta}{\cos\theta - e^{i\theta}} = \alpha\frac{\frac{e^{i\theta}-e^{-i\theta}}{2i}}{\frac{e^{i\theta}+e^{-i\theta}}{2}-e^{i\theta}} = -i\alpha\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}-2e^{i\theta}} = -i\alpha\frac{e^{i\theta}-e^{-i\theta}}{-e^{i\theta}+e^{-i\theta}} = i\alpha.$

Written in vector form:

$$\mathbf{x} = \alpha\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = \text{span}\{\mathbf{v}_1\}$$

For eigenvalue $\lambda_2 = e^{-i\theta}$ we find, after an almost identical procedure:

$$\mathbf{x} = \alpha\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \text{span}\{\mathbf{v}_2\}$$

For eigenvalue $\lambda_3 = 1$:

$$(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{x} = \mathbf{0}$$
$$\begin{pmatrix} \cos\theta - 1 & -\sin\theta & 0 \\ \sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

This time, the third column has no pivot and so is our free variable, while the other two variables must be zero. So:

$$\mathbf{x} = \alpha\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \text{span}\{\mathbf{v}_3\}$$

Therefore, the eigenvectors and eigenvalues are:

Eigenvector	Eigenvalue
$(i, 1,0)^T$	$e^{i\theta}$
$(-i, 1,0)^T$	$e^{-i\theta}$
$(0,0,1)^{\mathrm{T}}$	1

18) Find the eigenvectors and eigenvalues of the matrix $\begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}$.

Solution: First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\begin{vmatrix} 5-\lambda & -2 \\ 2 & 0-\lambda \end{vmatrix} = (5-\lambda)(-\lambda) + 4 = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0$$

Solutions are the eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 4$

Now we solve the associated eigenvector for each eigenvalue (when solving this, we always know the solution must have free parameters, because the determinant is zero):

For Eigenvalue $\lambda_1 = 1$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = 0$$
$$\begin{pmatrix} 5-1 & -2 \\ 2 & 0-1 \end{pmatrix}\mathbf{x} = \mathbf{0}$$

We solve by Gaussian elimination:

$$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

One free variable is $x_2 = \alpha$ and from the first row: $x_1 = \left(\frac{1}{2}\right)\alpha$. In vector form:

$$\mathbf{x} = \alpha\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right\} = \text{span}\{\mathbf{v}_1\}$$

The solution is always a span, we can take any scaled version as a valid eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For Eigenvalue $\lambda_2 = 4$:

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = 0$$
$$\begin{pmatrix} 5-4 & -2 \\ 2 & 0-4 \end{pmatrix}\mathbf{x} = \mathbf{0}$$

We solve by Gaussian elimination:

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

One free variable is $x_2 = \alpha$ and from the first row: $x_1 = 2\alpha$. In vector form:

$$\mathbf{x} = \alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} = \text{span}\{\mathbf{v}_2\}$$

We can take any scaled version as a valid eigenvector.
So we are finished. In summary, the eigenvectors and associated eigenvalues are:

$$\lambda_1 = 1 \;\to\; \mathbf{v}_1 = (1,2)$$
$$\lambda_2 = 4 \;\to\; \mathbf{v}_2 = (2,1)$$

This was not at all something we could have seen from intuition, as it is a very “weird” linear transformation. We can visually check it is true in a geometric software.

19) Calculate the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

Solution: First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues (be careful subtracting $\mathbf{I}$ when the matrix is written with a pre-factor like 1/3) :

$$\left|\frac{1}{3}\begin{pmatrix} 2-3\lambda & 2 & -1 \\ 2 & -1-3\lambda & 2 \\ -1 & 2 & 2-3\lambda \end{pmatrix}\right|$$
$$= \left(\frac{1}{3}\right)^3 [(2-3\lambda)^2(-1-3\lambda) - 4 - 4 - 4(2-3\lambda) - 4(2-3\lambda) - (-1-3\lambda)] = 0$$

$$(1/3)\,[-4 + 12\lambda - 9\lambda^2 - 12\lambda + 36\lambda^2 - 27\lambda^3 - 8 - 8 + 12\lambda - 8 + 12\lambda + 1 + 3\lambda] = 0$$
$$(1/3)(-27\lambda^3 + 27\lambda^2 + 27\lambda - 27) = 0$$
$$-\lambda^3 + \lambda^2 + \lambda - 1 = 0$$

$\lambda = 1$ is an obvious solution, so we can factorize out $(1-\lambda)$:

$$(1-\lambda)(\lambda^2 - 1) = 0$$
$$(1-\lambda)(1-\lambda)(-1-\lambda) = 0$$

So the eigenvalues are $\lambda_1 = \lambda_2 = 1$ (degenerate eigenvalues) and $\lambda_3 = -1$.

Now calculate the associated eigenvectors for each eigenvalue:

Case $\lambda_1 = \lambda_2 = 1$:

Solve the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = 1$. Use gaussian elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 2 & -4 & 2 & 0 \\ -1 & 2 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - R_1 \to R_3]{R_2 + 2R_1 \to R_2} \left(\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore we have two free variables (two columns without pivot) and we can solve the eigenvector:

$$x_2 = \alpha, x_3 = \beta, x_1 = 2x_2 - x_3 = 2\alpha - \beta$$

$$\mathbf{x} = \alpha\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

The eigenvalue was a double degenerate eigenvalue, and we obtained two associated eigenvectors.

(two degrees of freedom, which means two eigenvectors, but they are not unique, as any two linearly independent vectors contained in the plane could be used as pairs of eigenvectors)

Case $\lambda_3 = -1$:

Solve the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = -1$. Use gaussian elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} 5 & 2 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ -1 & 2 & 5 & 0 \end{array}\right) \xrightarrow[R_3 + (1/5)R_1 \to R_3]{R_2 - (2/5)R_1 \to R_2} \left(\begin{array}{ccc|c} 5 & 2 & -1 & 0 \\ 0 & 6/5 & 12/5 & 0 \\ 0 & 12/5 & 24/5 & 0 \end{array}\right) \xrightarrow[(5/6)R_2 \to R_2]{R_3 - 2R_2 \to R_3} \left(\begin{array}{ccc|c} 5 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore we have one free variable (third column without pivot) and we can solve by inverse substitution:

$$x_3 = \alpha, \;\; x_2 = -2x_3 = -2\alpha, \;\; x_1 = \tfrac{1}{5}(x_3 - 2x_2) = \tfrac{1}{5}(\alpha + 4\alpha) = \alpha$$

$$\mathbf{x} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\} = \mathrm{span}\{\mathbf{v}_3\}$$

So, in summary:

$\lambda_{1,2} = 1$ (degenerate eigenvalue) $\rightarrow$ $\mathbf{v}_{1,2} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ [or any other basis of that same plane]

$\lambda_3 = -1 \;\rightarrow\; \mathbf{v_3} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

With this information, and noticing that $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3$, we see that the eigenvalue 1 is associated with a plane, and the eigenvalue -1 is associated with the normal to the plane. Therefore, the linear transformation corresponds to a mirror symmetry on the plane given by $\mathbf{r} \cdot \mathbf{v}_3 = 0$.

20) Calculate the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Solution: First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues

$$p(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = (0 - \lambda)(0 - \lambda)$$

Solutions: $\lambda_{1,2} = 0$ (double degenerate eigenvalue)

Now calculate the associated eigenvectors by solving $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = 0$:

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Free variable (first column has no pivot) $x_1 = \alpha,$ and $x_2 = 0$

Therefore: $\mathbf{x} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ (one single eigenvector)

So in summary: $\lambda_{1,2} = 0 \;\;\rightarrow\;\; \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Table with selection of previous examples:

Matrix	$\begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
Charactersitic polynomial $p(\lambda)$	$(1-\lambda)\cdot(4-\lambda)$	$(1-\lambda)\cdot(1-\lambda)\cdot(-1-\lambda)$	$(0-\lambda)\cdot(0-\lambda)$
Eigenvalues λ_i (algebraic multiplicity)	$\lambda_1 = 1$ (1) $\lambda_2 = 4$ (1)	$\lambda_{1,2} = 1$ (2) $\lambda_3 = -1$ (1)	$\lambda_{1,2} = 0$ (2)
Eigenvectors $\mathbf{v}_i$ (geometric multiplicity)	$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (1) $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (1)	$\mathbf{v}_{1,2} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ (2) $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ (1)	$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (1)

Important theorems about eigenvectors/eigenvalues (as illustrated with the examples above):

- The characteristic polynomial always has N complex roots, and therefore N eigenvalues, but some of the roots can be repeated roots: these are called **degenerate eigenvalues**.
    - The number of times which a degenerate eigenvalue is repeated is called **algebraic multiplicity** of the eigenvalue.
    - The number of independent eigenvectors obtained from a given eigenvalue is called the **geometric multiplicity** of the eigenvalue.

  For each eigenvalue: $\boxed{1 \leq \textbf{Geometric multiplicity} \leq \textbf{Algebraic multiplicity}}$

  This means that an eigenvalue with algebraic multiplicity m can give rise to anything between 1 and m eigenvectors.

- **Eigenvalues which are different always give rise to linearly independent eigenvectors.**

Matrix $\mathbf{A}_{N\times N}$ has N different (non-degenerate) eigenvalues	$\Longrightarrow$ / $\not\Longleftarrow$	$\mathbf{A}$ has exactly N independent eigenvectors (an eigenbasis)	$\Longleftrightarrow$	$\mathbf{A}$ is diagonalizable

- The **determinant of a matrix is equal to the product of all its eigenvalues** (counting degenerate eigenvalues m times, where m is their multiplicity).
- The **trace of a matrix is equal to the sum of all its eigenvalues** (again counting degenerate eigenvalues m times)

- The **eigenvalues of a triangular matrix are exactly the elements of the diagonal**.

Some proofs of the above theorems:

• **If all eigenvalues are different, then all eigenvectors are linearly independent:**

Consider eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ for $i = 1,2, \dots, p$. Remember that the eigenvectors are linearly independent if and only if $\mathbf{w} = \left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p\right) = 0$ has only the trivial solution $c_i = 0$.

Let's prove that $c_1 = 0$ (later we will prove the same for $c_2, c_3, \dots$). Consider the following matrix product:

$$\mathbf{M}_1 = (\mathbf{A} - \lambda_2\mathbf{I})(\mathbf{A} - \lambda_3\mathbf{I}) \cdots (\mathbf{A} - \lambda_p\mathbf{I})$$

Since matrices $(\mathbf{A} - \lambda_i\mathbf{I})$ and $\left(\mathbf{A} - \lambda_j\mathbf{I}\right)$ commute [check it], the above matrix product can always be reordered to move any desired $(\mathbf{A} - \lambda_i\mathbf{I})$ term to the right of the product. Now consider multiplying this matrix $\mathbf{M}_1$ times the arbitrary linear combination of the eigenvectors:

$$\mathbf{M}_1\mathbf{w} = \mathbf{M}_1(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p)$$

Notice that $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_j = \mathbf{A}\mathbf{v}_j - \lambda_i\mathbf{v}_j = \left(\lambda_j - \lambda_i\right)\mathbf{v}_j$. Since we can always reorder the products on $\mathbf{M}$ we can make sure that each eigenvector is first multiplied by its corresponding term $\left(\mathbf{A} - \lambda_j\mathbf{I}\right)\mathbf{v}_j = \left(\lambda_j - \lambda_j\right)\mathbf{v}_j = 0$. So all terms go to zero except the $\mathbf{v}_1$ term because $\mathbf{M}_1$ does not include $(\mathbf{A} - \lambda_1\mathbf{I})$:

$$\mathbf{M}_1\mathbf{w} = c_1\mathbf{M}_1\mathbf{v}_1$$

But applying the property $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_1 = (\lambda_1 - \lambda_i)\mathbf{v}_1$ successively for each term in $\mathbf{M}_1$, we get that:

$$\mathbf{M}_1\mathbf{w} = c_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots \left(\lambda_1 - \lambda_p\right)\mathbf{v}_1$$

Clearly, if all eigenvalues are different, the only way to have $\mathbf{w} = 0$ is if $c_1 = 0$.

We can prove the same for all other c_k by using $\mathbf{M}_k = \prod_{i \neq k}(\mathbf{A} - \lambda_i\mathbf{I})$.

Therefore, all $c_k = 0$ for $\mathbf{w} = 0$, and so the eigenvectors are linearly independent.

• The **determinant of a matrix is equal to the product of all its eigenvalues**:

Consider the characteristic polynomial:

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_N - \lambda)$$

And now consider the case when $\lambda = 0$:

$$\Longrightarrow \quad p(\lambda = 0) = \det \mathbf{A} = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_N$$

• **The eigenvalues of a triangular matrix are exactly the elements of the diagonal.**

To prove the last statement, realise that the determinant of a triangular matrix is equal to the product of the diagonal elements, and if $\mathbf{A}$ is triangular, then so is the matrix $(\mathbf{A} - \lambda\mathbf{I})$. Therefore: $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{NN} - \lambda)$. Therefore $\lambda_i = a_{ii}$.

D. DIAGONALIZATION OF A MATRIX USING ITS EIGENBASIS

Sometimes, we can use knowledge of the eigenvectors of a matrix to “diagonalize it”. This is best understood through examples.

21) Consider the previous example for anisotropic scaling:

$$\mathbf{M} = \begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}$$

Eigenvectors	**Eigenvalue**
$(2,1)^{\mathrm{T}}$	4
$(1,2)^{\mathrm{T}}$	1

Do a change of basis so that we use the eigenvectors as the basis. What do you notice?

$$\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\} \rightarrow \{\mathbf{v}_1 = (2,1), \mathbf{v}_2 = (1,2)\}$$

In this basis, the linear transformation should look very simple because it will simply scale the basis vectors. Let’s do it. Let’s compute the matrices which change the basis $\{\mathbf{v}_1, \mathbf{v}_2\} \leftrightarrow \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$.

$$\mathbf{A} = \begin{pmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ translates basis } \{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$$

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \text{ calculated by doing the inverse, translates basis } \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\} \rightarrow \{\mathbf{v}_1, \mathbf{v}_2\}$$

And now let’s convert our linear transformation $\mathbf{M}$ into the language of basis $\{\mathbf{v}_1, \mathbf{v}_2\}$

$$\mathbf{M}' = \mathbf{A}^{-1}\mathbf{M}\mathbf{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

Not surprisingly, using the eigenvectors as our basis, **the matrix is now diagonal**! And the **elements of the diagonal are simply given by the eigenvalues** associated to each of the eigenvectors.

22) Consider the matrix for projection onto the line $y = mx$

$$\mathbf{M} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$$

Eigenvectors	**Eigenvalue**
$(1, m)^{\mathrm{T}}$	1
$(-m, 1)^{\mathrm{T}}$	0

Change the basis of this matrix to use the eigenvectors as basis.

We begin by finding the matrices for change of basis:

$$\mathbf{A} = \begin{pmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \text{ translates basis } \{\mathbf{v}_1, \mathbf{v}_2\} \to \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$$

$$\mathbf{A}^{-1} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \text{ translates basis } \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\} \to \{\mathbf{v}_1, \mathbf{v}_2\}$$

Therefore we can convert our linear transformation $\mathbf{M}$ into the language of basis $\{\mathbf{v}_1, \mathbf{v}_2\}$

$$\mathbf{M}' = \mathbf{A}^{-1}\mathbf{M}\mathbf{A} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}\frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$$

Let's do this in steps:

$$\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} = \begin{pmatrix} 1+m^2 & 0 \\ m+m^3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}\begin{pmatrix} 1+m^2 & 0 \\ m+m^3 & 0 \end{pmatrix} = \begin{pmatrix} 1+2m^2+m^4 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\frac{1}{1+m^2}\frac{1}{1+m^2}\begin{pmatrix} 1+2m^2+m^4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore:

$$\mathbf{M}' = \mathbf{A}^{-1}\mathbf{M}\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Again, this should not be surprising. In the basis $\left\{\begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix}\right\}$ the first basis vector is along the line of projection, and is left unchanged under transformation, while the second vector is orthogonal to the line of projection and is therefore projected to zero.

The matrix has been **diagonalized**, with the diagonal elements corresponding to the eigenvalues of each eigenvector.

DIAGONALIZABLE MATRICES

A matrix is said to be diagonalizable if you can find a **similar matrix** that is diagonal. That is, there exists a basis in which the transformation is represented by a diagonal matrix.

Diagonal matrices $\mathbf{D}$ represent a **scaling** (stretch/squeeze/flip/complex-phase-change) **along each of the N basis vector** directions. By definition $\mathbf{D}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ these directions must be eigenvectors $\mathbf{v}$ of the matrix, and the scaling factors will be the corresponding eigenvalues λ.

If a matrix is **similar** (technical term) to a diagonal matrix, it represents exactly the same transformation but expressed in a basis which does not coincide with the directions of scaling.

Diagonalization of matrices can be done whenever we can use the eigenvectors as a basis, **called an eigenbasis**. This means that there must be enough eigenvectors to span the whole space.

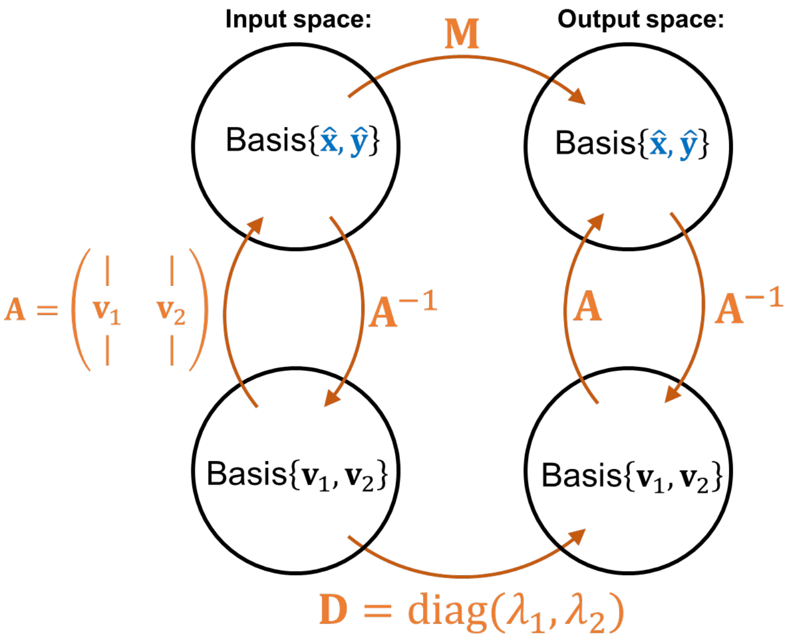
An $N \times N$ matrix is diagonalizable $\Longleftrightarrow$ it has N linearly independent eigenvectors (an eigenbasis)

When that condition holds, we can create an invertible change-of-basis matrix $\mathbf{A}$ to change the basis into the eigenbasis, and so we can write the matrix as a diagonal matrix: $\mathbf{D} = \mathbf{A}^{-1}\mathbf{M}\mathbf{A}$, where $\mathbf{A}$ is a matrix which contains the eigenvectors as its columns, and $\mathbf{D}$ is a diagonal matrix containing the eigenvalues as the main diagonal. This means that a diagonalizable matrix can be **completely defined** via its eigenvectors and eigenvalues as follows:

$$\mathbf{M} = \mathbf{A}\mathbf{D}\mathbf{A}^{-1}$$

$$\mathbf{M} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_N \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix} \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_N \\ | & | & & | \end{pmatrix}^{-1}$$

where $\mathbf{v}_i$ are the N eigenvectors of $\mathbf{M}$, each with corresponding eigenvalue λ_i.



Since $\mathbf{D}$ is diagonal, it is obvious that $\det \mathbf{D} = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_N$ is the product of the eigenvectors. But we know that $\det \mathbf{M} = \det \mathbf{D}$ because they represent the same transformation, therefore:

The determinant of an $N \times N$ matrix is equal to the product of its N eigenvalues (some may be repeated). The trace of an $N \times N$ matrix is equal to the sum of its N eigenvalues.

These properties are true even for matrices that are not diagonalizable, as mentioned earlier, but here we proved it for diagonalizable ones.

ADVANTAGES OF DIAGONAL MATRICES

Diagonal matrices are very useful because it is very easy to do calculations with them.

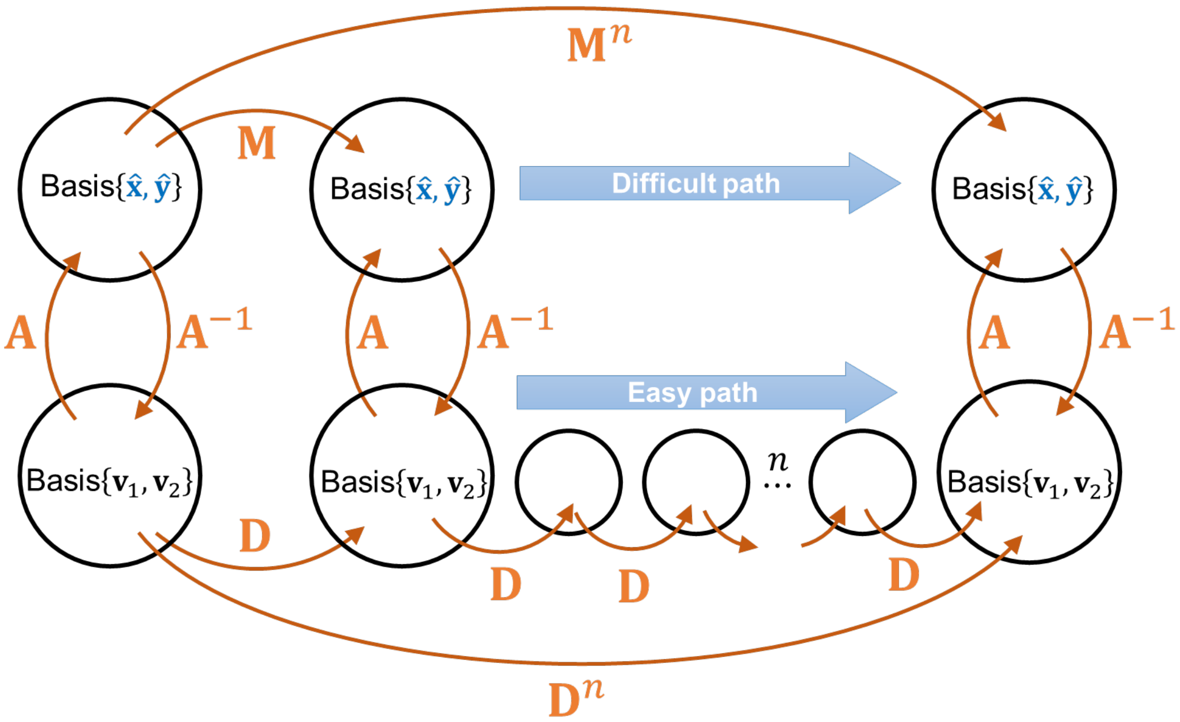
- The determinant of a diagonal matrix is equal to the product of all diagonal elements.
- The product of diagonal matrices is simply the product of the individual entries.

- Following the above, **two diagonal matrices always commute $\mathbf{AB} = \mathbf{BA}$**.
- Following the above, **raising a diagonal matrix to the n-th power becomes trivial**:

$$\mathbf{D}^n = [\operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)]^n = \operatorname{diag}(\lambda_1^n, \lambda_2^n, \cdots, \lambda_N^n)$$

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix}^n = \begin{pmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N^n \end{pmatrix}$$

While raising a non-diagonal matrix to an n-th power $\mathbf{M}^n$ is a slow process. This suggests a fast-easy way to compute the n-th power of a matrix for huge values of n:



Therefore, we can compute the n-th power of a matrix by doing:

$$\mathbf{M}^n = \mathbf{A}\mathbf{D}^n\mathbf{A}^{-1}$$

23) Diagonalize matrix $\mathbf{A}$ (reflection with respect to the $y = x$ line) and hence calculate its n-th power $\mathbf{A}^n$:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solution:

Eigenvalues and eigenvectors of this matrix are, from a previous problem:

$$\lambda_1 = 1 \ \rightarrow \ \mathbf{v}_1 = (1,1)^T$$
$$\lambda_2 = -1 \ \rightarrow \ \mathbf{v}_2 = (-1,1)^T$$

The two eigenvectors are linearly independent and form an eigenbasis, so we can change the basis of matrix $\mathbf{A}$ to the eigenbasis. For that, we use the change-of-basis matrix, and its inverse (trivial for a 2×2 matrix):

$$\mathbf{S} = \begin{pmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \rightarrow \quad \mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Such that the matrix in the new basis is given by the similarity transformation

$$\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{D}$$

Which is, as we expected, a diagonal matrix with the eigenvectors as its diagonal elements.

Hence, the matrix can be diagonalized:

$$\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$
$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

This allows us to calculate the n-th power of the matrix:

$$\mathbf{A}^n = (\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})\cdots(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}) = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1}$$
$$\mathbf{A}^n = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^n\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$\mathbf{A}^n = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$\mathbf{A}^n = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ (-1)^{n+1} & (-1)^n \end{pmatrix}$$
$$\mathbf{A}^n = \frac{1}{2}\begin{pmatrix} 1+(-1)^{n+2} & 1+(-1)^{n+1} \\ 1+(-1)^{n+1} & 1+(-1)^n \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & n \text{ even} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & n \text{ odd} \end{cases}$$

i.e. the powers of $\mathbf{A}$ alternate between $\mathbf{A}$ and $\mathbf{I}$, which is what one expects for a reflection symmetry transformation!

Interesting note: the expression for $\mathbf{A}^n$ also works for $\mathbf{A}^{-1}$ when $n = -1$. Indeed, it tells us that $\mathbf{A}^{-1} = \mathbf{A}$, as one would expect for a mirror symmetry operation, to be its own inverse.

24) Compute the 1000-th power of matrix $\mathbf{M} = \begin{pmatrix} 5/2 & -1 \\ 1 & 0 \end{pmatrix}$

Knowing that the eigenvectors and eigenvalues are:

Eigenvectors	Eigenvalue
$(2,1)^{\mathrm{T}}$	2
$(1,2)^{\mathrm{T}}$	1/2

The diagonalization of the matrix can be made by expressing the transformation in the basis of the eigenvectors, as done in a previous problem:

$$\mathbf{D} = \mathbf{A}^{-1}\mathbf{M}\mathbf{A} = \underbrace{\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{\mathbf{A}^{-1}} \underbrace{\begin{pmatrix} \frac{5}{2} & -1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{M}} \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{\mathbf{A}} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

As expected, $\mathbf{D}$ contains the eigenvalues. Now, matrix $\mathbf{D}$ represents the same linear transformation as $\mathbf{M}$, but in a different basis. Therefore, the linear transformation that results from applying $\mathbf{M}$ successively n times is equivalent to applying $\mathbf{D}$ successively n times (in the eigenvector basis). Multiplying diagonal matrices by themselves is simply multiplying the coefficients of the diagonal, therefore:

$$\mathbf{D}^n = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{\boldsymbol{n}} = \begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix}$$

And the associated matrix in the original $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ basis is, simply:

$$\mathbf{M}^n = \mathbf{A}\mathbf{D}^n\mathbf{A}^{-1}$$

This can be proven: $\mathbf{M}^n = (\mathbf{A}\mathbf{D}\mathbf{A}^{-1})^n = \mathbf{A}\mathbf{D}\underbrace{\mathbf{A}^{-1}\mathbf{A}}_{\mathbf{I}}\mathbf{D}\underbrace{\mathbf{A}^{-1}\mathbf{A}}_{\mathbf{I}}\mathbf{D}\mathbf{A}^{-1}\cdots\mathbf{A}\mathbf{D}\mathbf{A}^{-1} = \mathbf{A}\mathbf{D}^n\mathbf{A}^{-1}$

$$= \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix}}_{\mathbf{D}^n} \underbrace{\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{\mathbf{A}^{-1}}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2^{n+1} & -2^n \\ -2^{-n} & 2^{-n+1} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2^{n+2} - 2^{-n} & -2^{n+1} + 2^{-n+1} \\ 2^{n+1} - 2^{-n+1} & -2^n + 2^{-n+2} \end{pmatrix}$$

So, the $n = 1000$ case results in:

$$\mathbf{M}^{1000} = \frac{1}{3}\begin{pmatrix} 2^{1002} - 2^{-1000} & -2^{1001} + 2^{-999} \\ 2^{1001} - 2^{-999} & -2^{1000} + 2^{-998} \end{pmatrix}$$

Which would have been so difficult to find directly in the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ basis.

One of the beauties of mathematics is noticing how everything comes together so nicely... Now that we have an expression for $\mathbf{M}^n$ I could ask, can we calculate the inverse by using $n = -1$?

Of course we can! Logically, the inverse of a diagonal matrix is just calculating the inverse of its diagonal entries. $\mathbf{M}^{-1} = \frac{1}{3}\begin{pmatrix} 2^1 - 2^1 & -2^0 + 2^2 \\ 2^0 - 2^2 & -2^{-1} + 2^3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 3 \\ -3 & 15/2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 5/2 \end{pmatrix}$

FUNCTIONS OF MATRICES

We now know how to calculate $\mathbf{M}^n$. With this we can do something fascinating.

We know that a function $f(x) \approx a_0 + a_1 x + a_2 x^2 + \cdots$ can be written as a Taylor expansion.

So we can then define $f(\mathbf{M}) = a_0 + a_1\mathbf{M} + a_2\mathbf{M}^2 + \cdots$ and we know exactly how to compute it. Therefore we can compute any function applied to $\mathbf{M}$.

For example, we can calculate $\sin\mathbf{M}$, $\cos\mathbf{M}$, $\exp\mathbf{M}$, etc. Amazing!

25) Calculate $\exp(\mathbf{M})$ for the matrix:

$$\mathbf{M} = \begin{pmatrix} 5/2 & -1 \\ 1 & 0 \end{pmatrix}$$

Solution: we know the Taylor expansion for the exponential function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Therefore, we can define:

$$e^{\mathbf{M}} = 1 + \mathbf{M} + \frac{\mathbf{M}^2}{2!} + \frac{\mathbf{M}^3}{3!} + \cdots$$

But instead of doing this calculation in the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ basis which would be so difficult, we can do it in the basis where $\mathbf{M}$ is diagonal. Therefore:

$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \rightarrow e^{\mathbf{D}} = 1 + \mathbf{D} + \frac{\mathbf{D}^2}{2!} + \frac{\mathbf{D}^3}{3!} + \cdots = \begin{pmatrix} 1 + 2 + \cdots & 0 \\ 0 & 1 + \frac{1}{2} + \cdots \end{pmatrix} = \begin{pmatrix} e^2 & 0 \\ 0 & e^{1/2} \end{pmatrix}$$

Switching back to the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ basis

$$(e^{\mathbf{M}}) = \mathbf{A}(e^{\mathbf{D}})\mathbf{A}^{-1} = \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} e^2 & 0 \\ 0 & e^{1/2} \end{pmatrix}}_{e^{\mathbf{D}}} \underbrace{\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{\mathbf{A}^{-1}} = \frac{1}{3}\begin{pmatrix} -\sqrt{e} + 4e^2 & -2(-\sqrt{e} + e^2) \\ 2(-\sqrt{e} + e^2) & 4\sqrt{e} - e^2 \end{pmatrix}$$

26) Calculate $\sqrt{\mathbf{M}}$, the square root of the matrix, knowing from a previous problem that

$$\mathbf{M}^n = \frac{1}{3}\begin{pmatrix} 2^{n+2} - 2^{-n} & -2^{n+1} + 2^{-n+1} \\ 2^{n+1} - 2^{-n+1} & -2^n + 2^{-n+2} \end{pmatrix}$$

Rather than going through the Taylor path, we simply use $n = 1/2$. This gives us

$$\sqrt{\mathbf{M}} = \mathbf{M}^{\frac{1}{2}} = \frac{1}{3}\begin{pmatrix} 2^{\frac{1}{2}+2} - 2^{-\frac{1}{2}} & -2^{\frac{1}{2}+1} + 2^{-\frac{1}{2}+1} \\ 2^{\frac{1}{2}+1} - 2^{-\frac{1}{2}+1} & -2^{\frac{1}{2}} + 2^{-\frac{1}{2}+2} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \frac{7}{\sqrt{2}} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$$

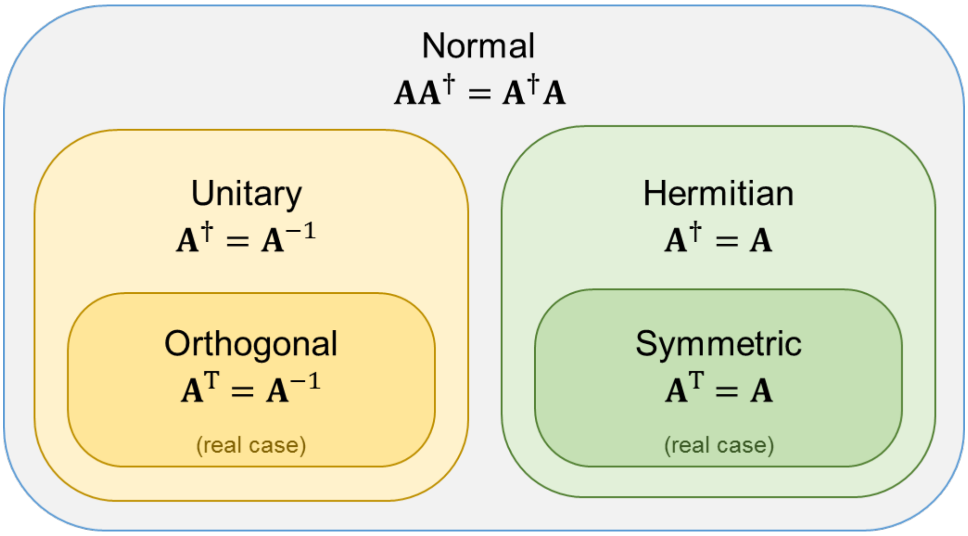
Notice how remarkable it is that, indeed, $\sqrt{\mathbf{M}}\,\sqrt{\mathbf{M}} = \mathbf{M}$

$$\frac{1}{3}\begin{pmatrix} \frac{7}{\sqrt{2}} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix} \frac{1}{3}\begin{pmatrix} \frac{7}{\sqrt{2}} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 5/2 & -1 \\ 1 & 0 \end{pmatrix}$$

Also notice that $\mathbf{M}^0 = \mathbf{I}$ and $\mathbf{M}^1 = \mathbf{M}$. In fact this is how I animate transformations in the interactive presentations! With a slider t, by showing the transformation $\mathbf{M}^t$ for $t \in [0,1]$.

E. EIGENVECTORS OF SPECIAL TYPES OF MATRICES

Special types of matrices have interesting properties in their eigenvectors. The following Venn diagram summarizes the special types of matrices most widely used:



Remember that this is a Venn diagram, therefore all unitary and all Hermitian matrices are normal. If the matrix is real, then $\mathbf{A}^\dagger = \mathbf{A}^\mathrm{T}$, so all real symmetric matrices are Hermitian, and all real orthogonal matrices are unitary. All properties of normal matrices apply to all the sub-types.

NORMAL MATRICES:

The following important theorem about the eigenvectors of normal matrices exists:

$\mathbf{A}$ is normal ($\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$)	$\Longrightarrow$	All eigenvectors of $\mathbf{A}$ can be chosen **orthogonal to each other** (even if there are some degenerate eigenvalues).	$\Rightarrow$	$\mathbf{A}$ has an **orthonormal eigenbasis spanning the input space** and is therefore unitarily diagonalizable: $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger$

(because $\mathbf{U}$ has orthonormal eigenvectors as its columns and is thus a unitary matrix $\mathbf{U}^{-1} = \mathbf{U}^\dagger$)

See book [*Riley, Hobson, Bence*] for proof of part of it (a general proof is out of the scope)

If all eigenvalues are different (non-degenerate) then the eigenvectors of a normal matrix are automatically orthogonal, not only independent as for any other matrix. If some eigenvalues are degenerate, a normal matrix ensures they will have a geometric multiplicity equal to the algebraic multiplicity, not lower: the eigenvectors for degenerate eigenvalues are not unique, they can be chosen arbitrarily to span the necessary eigenspace. The theorem above tells us that the eigenspace is always orthogonal to other eigenvectors, so the eigenvectors can be chosen to be orthogonal to all other eigenvectors.

HERMITIAN MATRICES:

Hermitian matrices inherit all the properties of normal matrices (copied below in shortened form), plus a new one; that **all eigenvalues are real**.

$\mathbf{A}$ is Hermitian ($\mathbf{A}^\dagger = \mathbf{A}$) $\Longrightarrow$ $\mathbf{A}$ is Normal ($\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$) $\Longrightarrow$ $\mathbf{A}$ has N orthonormal eigenvectors

$\mathbf{A}$ is Hermitian ($\mathbf{A}^\dagger = \mathbf{A}$) $\Longrightarrow$ All eigenvalues of $\mathbf{A}$ are real

In summary, Hermitian matrices have the "nicest" possible eigenvalues and eigenvectors: purely real eigenvalues and a complete basis of N orthonormal eigenvectors which spans the input space.

Of course, symmetric real matrices (which are the real counterpart to Hermitian matrices) share all these properties, replacing $\mathbf{A}^\dagger = \mathbf{A}^\mathrm{T}$.

Proof all eigenvalues of $\mathbf{A}$ are real: Assume $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for eigenvector $\mathbf{v}$ and eigenvalue λ. We can take Hermitian conjugate of both sides to obtain $(\mathbf{A}\mathbf{v})^\dagger = (\lambda\mathbf{v})^\dagger \to \mathbf{v}^\dagger\mathbf{A}^\dagger = \lambda^*\mathbf{v}^\dagger$. Multiply on the right by $\mathbf{v}$ to get $\mathbf{v}^\dagger\mathbf{A}^\dagger\mathbf{v} = \lambda^*\mathbf{v}^\dagger\mathbf{v}$, and since $\mathbf{A}^\dagger = \mathbf{A}$, we can say: $\mathbf{v}^\dagger\mathbf{A}^\dagger\mathbf{v} = \mathbf{v}^\dagger\mathbf{A}\mathbf{v} = \lambda\mathbf{v}^\dagger\mathbf{v}$. Comparing both results, we find that $\lambda^* = \lambda$. Therefore λ is real.

Proof $\mathbf{A}$ has N orthonormal eigenvectors: Assuming different eigenvalues (the general proof is longer). We assume (a) $\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ and (b) $\mathbf{A}\mathbf{v}_j = \lambda_j\mathbf{v}_j$. Now we take Hermitian conjugate of the first eq. and multiply right by $\mathbf{v}_j$ resulting in (a) $\mathbf{v}_i^\dagger\mathbf{A}^\dagger\mathbf{v}_j = \lambda_i^*\mathbf{v}_i^\dagger\mathbf{v}_j$. Next, we multiply the second eq. on the left by $\mathbf{v}_i^\dagger$ resulting in (b) $\mathbf{v}_i^\dagger\mathbf{A}\mathbf{v}_j = \lambda_j\mathbf{v}_i^\dagger\mathbf{v}_j$. Subtracting (a) and (b) we obtain: $\mathbf{v}_i^\dagger\mathbf{A}^\dagger\mathbf{v}_j - \mathbf{v}_i^\dagger\mathbf{A}\mathbf{v}_j = (\lambda_i^* - \lambda_j)\mathbf{v}_i^\dagger\mathbf{v}_j$ and since $\mathbf{A}^\dagger = \mathbf{A}$, the left hand side is zero, and $\lambda_i^* = \lambda_i$ because all λ are real, so we have: $(\lambda_i - \lambda_j)\mathbf{v}_i^\dagger\mathbf{v}_j = (\lambda_i - \lambda_j)\langle\mathbf{v}_i, \mathbf{v}_j\rangle = \mathbf{0}$. So, if $\lambda_i \neq \lambda_j$, then $\langle\mathbf{v}_i, \mathbf{v}_j\rangle = \mathbf{0}$ meaning they are orthogonal.

UNITARY MATRICES:

Unitary matrices inherit all the properties of normal matrices (copied below in shortened form), plus a new one; that all eigenvalues lie in the unit circle of the complex plane.

$\mathbf{A}$ is Unitary ($\mathbf{A}^\dagger = \mathbf{A}^{-1}$) $\Longrightarrow$ $\mathbf{A}$ is Normal ($\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$) $\Longrightarrow$ $\mathbf{A}$ has N orthonormal eigenvectors

$\mathbf{A}$ is Unitary ($\mathbf{A}^\dagger = \mathbf{A}^{-1}$) $\Longrightarrow$ All eigenvalues of $\mathbf{A}$ have unit modulus $|\lambda_k| = 1$ (i.e. $\lambda_k = e^{i\theta_k}$)

Proof: We proved earlier that unitary matrices preserve the norms of the vectors they transform: $||\mathbf{A}\mathbf{x}|| = ||\mathbf{x}||$. Therefore, this must of course include the eigenvectors: $||\mathbf{A}\mathbf{v}|| = |\lambda|||\mathbf{v}|| = ||\mathbf{v}||$.

Of course, orthogonal real matrices (which are the real counterpart to unitary matrices) share all these properties, replacing $\mathbf{A}^\dagger = \mathbf{A}^\mathrm{T}$.

27) Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & i & i \\ -i & 1 & 0 \\ -i & 0 & 1 \end{pmatrix}$$

First notice that **the matrix is Hermitian**, so even though it is a complex matrix, we know **it will have 3 real eigenvalues and 3 orthogonal eigenvectors**. First solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\begin{vmatrix} 0-\lambda & i & i \\ -i & 1-\lambda & 0 \\ -i & 0 & 1-\lambda \end{vmatrix} = \big((-\lambda)(1-\lambda)(1-\lambda) + 0 + 0 - 0 - (1-\lambda) - (1-\lambda)\big) = 0$$

$$-\lambda + 2\lambda^2 - \lambda^3 - 2 + 2\lambda = -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

By trial and error we can easily see that $\lambda = 1$ is a solution, as well as $\lambda = -1$. With that information we can factorize the characteristic polynomial completely and find the three eigenvalues:

$$(1-\lambda)(-1-\lambda)(2-\lambda) = 0$$

Calculate eigenvectors for $\lambda_1 = 1$. For this we solve the equation $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = 0$. Let's do Gauss elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} -1 & i & i & 0 \\ -i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_3 - iR_1 \to R_3]{R_2 - iR_1 \to R_2} \left(\begin{array}{ccc|c} -1 & i & i & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & i & i & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have one degree of freedom (the third column has no pivot). Therefore: 3rd row: $x_3 = \alpha$. 2nd row: $x_2 = -\alpha$. 1st row: $x_1 = 0$.

$$\mathbf{x} = \text{span}\{(0,1,-1)^T\} \to \mathbf{v}_1 = (0,1,-1)^T \text{ or any multiple}$$

Calculate eigenvectors for $\lambda_2 = -1$. For this we solve the equation $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = 0$. Let's do Gauss elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & i & i & 0 \\ -i & 2 & 0 & 0 \\ -i & 0 & 2 & 0 \end{array}\right) \xrightarrow[R_3 + iR_1 \to R_3]{R_2 + iR_1 \to R_2} \left(\begin{array}{ccc|c} 1 & i & i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & i & i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have one degree of freedom (the third column has no pivot). Therefore: 3rd row: $x_3 = \alpha$. 2nd row: $x_2 = \alpha$. 1st row: $x_1 = -2i\alpha$.

$$\mathbf{x} = \text{span}\{(-2i,1,1)^T\} \to \mathbf{v}_2 = (-2i,1,1)^T \text{ or any multiple}$$

Calculate eigenvectors for $\lambda_3 = 2$. For this we solve the equation $(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{x} = 0$. Let's do Gauss elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} -2 & i & i & 0 \\ -i & -1 & 0 & 0 \\ -i & 0 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - (i/2)R_1 \to R_3]{R_2 - (i/2)\,R_1 \to R_2} \left(\begin{array}{ccc|c} -2 & i & i & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & -1/2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -2 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have one degree of freedom (the third column has no pivot). Therefore: 3rd row: $x_3 = \alpha$. 2nd row: $x_2 = \alpha$. 1st row: $x_1 = (1/2)(x_2 i + x_3 i) = i$.

$$\mathbf{x} = \text{span}\{(i,1,1)^T\} \to \mathbf{v}_3 = (i,1,1)^T \text{ or any multiple}$$

Which are indeed three orthogonal vectors

28) Calculate the 100-th power $\mathbf{A}^{100}$ of the matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & i & 0 \\ -i & 1 & i \\ 0 & -i & 0 \end{pmatrix}$$

In order to calculate such a high power, the only practical method is to diagonalize the matrix:

$$\mathbf{A} = \mathbf{SDS}^{-1}$$

Such that:

$$\mathbf{A}^n = \mathbf{SD}^n\mathbf{S}^{-1}$$

where $\mathbf{D}$ is a diagonal matrix containing the eigenvalues, whose n-th power is trivial to calculate:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \to \mathbf{D}^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}$$

and where $\mathbf{S}$ is a matrix containing the eigenvectors:

$$\mathbf{S} = \begin{pmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{pmatrix}$$

Therefore, our first task is to calculate the eigenvalues and eigenvectors of the matrix.

For this, first notice that **the matrix is Hermitian**, so even though it is a complex matrix, we know **it will have 3 real eigenvalues and 3 orthogonal eigenvectors**.

Therefore, the eigenvectors form an **orthonormal eigenbasis** $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3\}$ which means that the matrix $\mathbf{S}$ must be a unitary matrix, which in turn means that calculating the inverse becomes trivial:

$$\mathbf{S} = \begin{pmatrix} | & | & | \\ \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 \\ | & | & | \end{pmatrix} \text{ with } \{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3\} \text{ orthonormal basis} \Leftrightarrow \mathbf{S} \text{ is a unitary matrix} \Leftrightarrow \mathbf{S}^{-1} = \mathbf{S}^\dagger$$

$$\mathbf{S}^{-1} = \begin{pmatrix} - & \hat{\mathbf{v}}_1^* & - \\ - & \hat{\mathbf{v}}_2^* & - \\ - & \hat{\mathbf{v}}_3^* & - \end{pmatrix}$$

Knowing all this, let's start by finding the eigenvalues and eigenvectors of $\mathbf{A}$.

Calculating the determinant and trace of the matrix $\mathbf{A}$ is quick: $\det \mathbf{A} = 0 + 0 + 0 - 0 - 0 - 0 = 0$ using the diagonals method, and $\operatorname{tr} \mathbf{A} = 1$ by adding the elements of the diagonal. This quick calculation can help us finding the eigenvalues, since now we know that $\det \mathbf{A} = \lambda_1\lambda_2\lambda_3 = 0$ and $\operatorname{tr} \mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3 = 1$. From this we can deduce that $\lambda_1 = 0$ and $\lambda_2 = 1 - \lambda_3$.

Now, solve the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to find eigenvalues:

$$\begin{vmatrix} 0-\lambda & i & 0 \\ -i & 1-\lambda & i \\ 0 & -i & 0-\lambda \end{vmatrix} = \big((-\lambda)(1-\lambda)(-\lambda) + 0 + 0 - 0 - (i)(-i)(-\lambda) - (-\lambda)(i)(-i)\big) = 0$$

$$\lambda^2 - \lambda^3 + \lambda + \lambda = 0$$
$$-\lambda^3 + \lambda^2 + 2\lambda = 0$$
$$-\lambda(\lambda^2 - \lambda - 2) = 0$$
$$-\lambda(\lambda + 1)(\lambda - 1) = 0$$

Therefore, the three roots are the three eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = 2$. Quick check, this matches the expected $\det \mathbf{A} = \lambda_1\lambda_2\lambda_3 = 0$ and $\operatorname{tr} \mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3 = 1$. Also, since $\mathbf{A}$ was Hermitian we know that its eigenvalues are all real.

Calculate eigenvectors for $\lambda_1 = 0$. For this we solve the equation $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = 0$. Let's do Gauss elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & i & 0 & 0 \\ -i & 1 & i & 0 \\ 0 & -i & 0 & 0 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_1]{R_3 + R_1 \to R_3} \left(\begin{array}{ccc|c} -i & 1 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have one degree of freedom (the third column has no pivot). Therefore: 3rd row: $x_3 = \alpha$. 2nd row: $ix_2 = 0 \to x_2 = 0$. 1st row: $-ix_1 + i\alpha = 0 \to x_1 = \alpha$.

$$\mathbf{x} = \alpha(1,0,1)^T = \operatorname{span}\{(1,0,1)^T\} \to \mathbf{v}_1 = (1,0,1)^T \text{ or any multiple}$$

Calculate eigenvectors for $\lambda_2 = -1$. For this we solve the equation $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = 0$ (we must subtract -1, i.e. add 1, to the main diagonal). Let's do Gauss elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ -i & 2 & i & 0 \\ 0 & -i & 1 & 0 \end{array}\right) \xrightarrow{R_2 + (-i)R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \end{array}\right) \xrightarrow{R_3 + (i)R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have one degree of freedom (the third column has no pivot). Therefore: 3rd row: $x_3 = \alpha$. 2nd row: $x_2 + i\alpha = 0 \to x_2 = -i\alpha$. 1st row: $x_1 + ix_2 = 0 \to x_1 = -ix_2 = -\alpha$.

$$\mathbf{x} = \alpha(-1,-i,1)^{\mathrm{T}} = \operatorname{span}\{(-1,-i,1)^{\mathrm{T}}\} \to \mathbf{v}_2 = (-1,-i,1)^{\mathrm{T}} \text{ or any multiple}$$

Calculate eigenvectors for $\lambda_3 = 2$. For this we solve the equation $(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{x} = 0$. Let's do Gauss elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} -2 & i & 0 & 0 \\ -i & -1 & i & 0 \\ 0 & -i & -2 & 0 \end{array}\right) \xrightarrow{R_2 + (-i/2)\, R_1 \to R_2} \left(\begin{array}{ccc|c} -2 & i & 0 & 0 \\ 0 & -1/2 & i & 0 \\ 0 & -i & -2 & 0 \end{array}\right) \xrightarrow{R_3 + (-2i)R_2 \to R_3} \left(\begin{array}{ccc|c} -2 & i & 0 & 0 \\ 0 & -1/2 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We have one degree of freedom (the third column has no pivot). Therefore: 3rd row: $x_3 = \alpha$. 2nd row: $x_2 = 2ix_3 = 2i\alpha$. 1st row: $2x_1 = ix_2 \to x_1 = -\alpha$.

$$\mathbf{x} = \alpha(-1,2i,1)^{\mathrm{T}} = \operatorname{span}\{(-1,2i,1)^T\} \to \mathbf{v}_3 = (-1,2i,1)^T \text{ or any multiple}$$

Which are indeed three orthogonal vectors:

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= (1)(-1)^* + 0 + (1)(1)^* = 0 \\ \langle \mathbf{v}_1, \mathbf{v}_3 \rangle &= (1)(-1)^* + 0 + (1)(1)^* = 0 \\ \langle \mathbf{v}_2, \mathbf{v}_3 \rangle &= (-1)(-1)^* + (-i)(2i)^* + (1)(1)^* = 1 - 2 + 1 = 0 \end{aligned}$$

We would like to have an **orthonormal eigenbasis**. For that, we can simply normalize each eigenvector by its norm so that they all become unit vectors:

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \hat{\mathbf{v}}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}, \qquad \hat{\mathbf{v}}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2i \\ 1 \end{pmatrix}$$

Therefore, we can now construct the unitary change of basis matrix:

$$\mathbf{S} = \begin{pmatrix} | & | & | \\ \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -i\sqrt{2} & 2i \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix}$$

And we know its inverse must be equal to its Hermitian conjugate (since $\mathbf{S}$ is unitary):

$$\mathbf{S}^{-1} = \mathbf{S}^{\dagger} = \begin{pmatrix} - & \hat{\mathbf{v}}_1^* & - \\ - & \hat{\mathbf{v}}_2^* & - \\ - & \hat{\mathbf{v}}_3^* & - \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & i\sqrt{2} & \sqrt{2} \\ -1 & -2i & 1 \end{pmatrix}$$

The diagonal matrix $\mathbf{D}$ simply contains the eigenvalues (please note, the ordering of eigenvalues is arbitrary, however, we must be consistent: the first element in the diagonal must be the eigenvalue for the first column vector in matrix $\mathbf{S}$, and so on…)

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

At this point, we can check, doing matrix multiplication, that, indeed:

$$\mathbf{A} = \mathbf{SDS}^{-1}$$

$$\begin{pmatrix} 0 & i & 0 \\ -i & 1 & i \\ 0 & -i & 0 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -i\sqrt{2} & 2i \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & i\sqrt{2} & \sqrt{2} \\ -1 & -2i & 1 \end{pmatrix}$$

So we have successfully diagonalized the matrix! Now it's very easy to calculate its 100-th power:

$$\mathbf{A}^{100} = (\mathbf{SDS}^{-1})(\mathbf{SDS}^{-1})(\mathbf{SDS}^{-1})\cdots(\mathbf{SDS}^{-1}) = \mathbf{SD}^{100}\mathbf{S}^{-1}$$

$$\mathbf{A}^{100} = \frac{1}{6}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -i\sqrt{2} & 2i \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^{100} & 0 \\ 0 & 0 & 2^{100} \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & i\sqrt{2} & \sqrt{2} \\ -1 & -2i & 1 \end{pmatrix}$$

Knowing that $(-1)^{100} = 1$, we now do matrix multiplication:

$$\mathbf{A}^{100} = \frac{1}{6}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -i\sqrt{2} & 2i \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & i\sqrt{2} & \sqrt{2} \\ -2^{100} & -2^{101}i & 2^{100} \end{pmatrix}$$
$$= \frac{1}{6}\begin{pmatrix} 2 + 2^{100} & -2i + 2^{101}i & -2 - 2^{100} \\ 2i - 2^{101}i & 2 + 2^{102} & -2i + 2^{101}i \\ -2 - 2^{100} & 2i - 2^{101}i & 2 + 2^{100} \end{pmatrix}$$

which is the requested answer.

F. SIMULTANEOUS EIGENVECTORS AND COMMUTATION OF MATRICES

Theorem:

Matrices $\mathbf{A}$ and $\mathbf{B}$ are **simultaneously diagonalizable** (i.e. can be diagonalized in the **same** basis)	$\Longrightarrow$	Matrices $\mathbf{A}$ and $\mathbf{B}$ commute $\mathbf{AB} = \mathbf{BA}$

Proof. If we can construct the shared change-of-basis matrix $\mathbf{V}$ which diagonalizes both $\mathbf{A}$ and $\mathbf{B}$: $\mathbf{D}_A = \mathbf{V}^{-1}\mathbf{AV}$ and $\mathbf{D}_B = \mathbf{V}^{-1}\mathbf{BV}$ (definition of "simultaneously diagonalizable"). Then the product of the diagonalized versions obviously commutes (as they are diagonal) and therefore so do $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{AB} = (\mathbf{VD}_{\mathrm{A}}\mathbf{V}^{-1})(\mathbf{VD}_{\mathrm{B}}\mathbf{V}^{-1}) = \mathbf{VD}_{\mathrm{A}}\mathbf{D}_{\mathrm{B}}\mathbf{V}^{-1} = \mathbf{VD}_{\mathrm{B}}\mathbf{D}_{\mathrm{A}}\mathbf{V}^{-1} = (\mathbf{VD}_{\mathrm{B}}\mathbf{V}^{-1})(\mathbf{VD}_{\mathrm{A}}\mathbf{V}^{-1}) = \mathbf{BA}$$

This makes intuitive sense, because if the linear transformations commute, they do so regardless of the basis.

And we can make the theorem work **both ways** if we assume that matrices are normal matrices:

If both matrices $\mathbf{A}$ and $\mathbf{B}$ are **normal** matrices (includes Hermitian and unitary ones), then:

$\mathbf{A}$ and $\mathbf{B}$ have **the same eigenvectors**	$\Longleftrightarrow$	$\mathbf{A}$ and $\mathbf{B}$ are **simultaneously diagonalizable**	$\Longleftrightarrow$	Matrices $\mathbf{A}$ and $\mathbf{B}$ **commute** $\mathbf{AB} = \mathbf{BA}$

Let's prove this only in the simpler case when both $\mathbf{A}$ and $\mathbf{B}$ have N different eigenvalues*:

The first relation is simple: if they have the same eigenbasis, they can both be diagonalized with it.

The second relation: The proof in one direction was done above. We now need to prove the other direction:

Assumptions are that $\mathbf{AB} = \mathbf{BA}$ and that $\mathbf{A}$ and $\mathbf{B}$ have N different eigenvectors $\mathbf{A}\mathbf{v}_{Ai} = \lambda_i \mathbf{v}_{Ai}$ and $\mathbf{B}\mathbf{v}_{Bi} = \mu_i \mathbf{v}_{Bi}$.

Our task is to prove that they must have the same eigenvectors $\mathbf{v}_{Ai} = \mathbf{v}_{Bi}$ even though they might have different eigenvalues. Take the product $\mathbf{AB}$ and multiply by the right with $\mathbf{v}_{Ai}$, since the two matrices commute:

$$\mathbf{AB}\mathbf{v}_{Ai} = \mathbf{BA}\mathbf{v}_{Ai} = \mathbf{B}\lambda_i\mathbf{v}_{Ai} = \lambda_i\mathbf{B}\mathbf{v}_{Ai}$$

Therefore, equating the first and last of these: $\mathbf{A}(\mathbf{B}\mathbf{v}_{Ai}) = \lambda_i(\mathbf{B}\mathbf{v}_{Ai})$, which means that $\mathbf{B}\mathbf{v}_{Ai}$ is an eigenvector of $\mathbf{A}$ associated to eigenvalue λ_i, just as $\mathbf{v}_{Ai}$ was. But eigenvector solutions associated to the same single λ_i are unique to within a scale factor, therefore it must be true that $\mathbf{B}\mathbf{v}_{Ai} = \mu_i\mathbf{v}_{Ai}$, which means that $\mathbf{v}_{Ai}$ is an eigenvector of $\mathbf{B}$ so $\mathbf{v}_{Ai} = \mathbf{v}_{Bi}$.

*the proof can be generalised to the condition that there are some degenerate eigenvalues, but being normal still have N independent eigenvectors. As long as by taking a linear combination of these vectors, one set of joint eigenvectors can be found between $\mathbf{A}$ and $\mathbf{B}$.

This theorem is fundamental to the foundations of quantum mechanics and the uncertainty principle.

Intuitively, the two transformations are simply scaling the vector space along the **same** directions, so we can simply scale each direction individually and the order is not important (i.e. diagonal matrices).

G. APPLICATIONS OF EIGENVECTORS IN PHYSICS

The notion of eigenvectors and eigenvalues is of huge importance in many areas of physics. Quantum mechanics is almost entirely based on it.

We include here a simple example. You can find more examples in Prof. Lev Kantorovich's 5CCP2255 Notes (page 39).

29) 3D motion of a particle in an electromagnetic field: find the trajectory of the particle as a function of time $\mathbf{r}(t)$ and its velocity $\mathbf{v}(t)$, given initial conditions $\mathbf{r}(0)$ and $\mathbf{v}(0)$.

Consider a particle of charge q and mass m in a constant magnetic field $\mathbf{B}$. The equation of motion, given by combining Newton's equation with the Lorentz force, is:

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} = q(\mathbf{v}\times\mathbf{B})$$

This, being a vector equation, can be written explicitly component by component, which can then be written as a matrix equation:

$$\begin{cases} m\dot{v_1} = q(v_2B_3 - v_3B_2) \\ m\dot{v_2} = q(v_3B_1 - v_1B_3) \\ m\dot{v_3} = q(v_1B_2 - v_2B_1) \end{cases} \quad \rightarrow \quad m\begin{pmatrix} \dot{v_1} \\ \dot{v_2} \\ \dot{v_3} \end{pmatrix} = q\begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

As a matrix equation, this is:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{G}\mathbf{v}$$

with:

$$\mathbf{G} = \frac{q}{m}\begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}$$

To solve this equation, we use the ansatz $\mathbf{v} = \mathbf{u}e^{\alpha t}$ where $\mathbf{u}$ and α are unknown. This converts the matrix equation into a known eigenvalue problem $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}e^{\alpha t}) = \mathbf{G}\mathbf{u}e^{\alpha t} \rightarrow \alpha\mathbf{u}e^{\alpha t} = \mathbf{G}\mathbf{u}e^{\alpha t}$

$$\alpha\mathbf{u} = \mathbf{G}\mathbf{u}$$

Therefore, we simply need to find eigenvalues α_i and eigenvectors $\mathbf{u}_i$ as possible solutions. The general solution is a linear combination:

$$\mathbf{v}(\mathrm{t}) = c_1\mathbf{u}_1e^{\alpha_1 t} + c_2\mathbf{u}_2e^{\alpha_2 t} + c_3\mathbf{u}_3e^{\alpha_3 t}$$

The undefined constants can be obtained from the initial conditions.

For example, let's solve a simple case: assume $\mathbf{B} = B\hat{\mathbf{z}}$. The matrix $\mathbf{G}$ becomes:

$$\mathbf{G} = \frac{q}{m}\begin{pmatrix} 0 & B & 0 \\ -B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Whose characteristic polynomial is:

$$\det(\mathbf{G} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & \omega & 0 \\ -\omega & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 + \omega^2) = -\lambda(\lambda - i\omega)(\lambda + i\omega) = 0$$

The eigenvalues are: $\lambda = 0, i\omega, -i\omega$.

The eigenvectors are (normalizing them to be unit vectors for ease of use): $\mathbf{u}_1 = (0,0,1)^T, \mathbf{u}_2 = \frac{1}{\sqrt{2}}(1, i, 0)^T$ and $\mathbf{u}_3 = \frac{1}{\sqrt{2}}(1, -i, 0)^T$. Therefore, the general solution to the equation of motion is:

$$\mathbf{v}(t) = c_1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{c_2}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}e^{i\omega t} + \frac{c_3}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}e^{-i\omega t}$$

Complex velocities? Don't worry, the last two terms will combine to form sines and cosines once we find the values of c_2 and c_3 from the initial conditions. For example, let's use a general initial velocity $\mathbf{v}(0) = (0, v_\perp, v_\parallel)$:

$$\mathbf{v}(t = 0) = c_1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{c_2}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}e^{0} + \frac{c_3}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}e^{0} = \begin{pmatrix} 0 \\ v_\perp \\ v_\parallel \end{pmatrix}$$

We find: $c_1 = v_\parallel$ and $c_2 = -c_3 = -\frac{i}{\sqrt{2}}v_\perp$. Substituting these into the equation of motion we get:

$$\mathbf{v}(t) = v_\parallel\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{iv_\perp}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}e^{i\omega t} + \frac{iv_\perp}{2}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}e^{-i\omega t}$$

$$\mathbf{v}(t) = \begin{pmatrix} \frac{v_\perp}{2}\left(-ie^{i\omega t} + ie^{-i\omega t}\right) \\ \frac{v_\perp}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \\ v_\parallel \end{pmatrix} = \begin{pmatrix} v_\perp \sin\omega t \\ v_\perp \cos\omega t \\ v_\parallel \end{pmatrix}$$

Interestingly, notice that the kinetic energy of the particle $K(t) = \frac{1}{2}m\|\mathbf{v}\|^2 =$ constant, as it is well known that the magnetic field does not do any work on the particle. The particle trajectory can be obtained by integrating the velocity with respect to time:

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(t')\mathrm{d}t' = \begin{pmatrix} -\frac{v_\perp}{\omega}\cos\omega t + r_1 \\ \frac{v_\perp}{\omega}\sin\omega t + r_2 \\ v_\parallel t + r_3 \end{pmatrix}$$

This corresponds to a helical trajectory! where r_i are the integration constants that can be obtained from the initial position $\mathbf{r}(0)$.

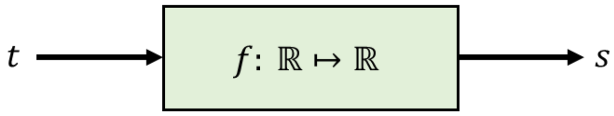
3. FUNCTIONS OF HIGHER DIMENSIONS

Most of your past study of mathematics has involved functions $f(x)$, where one variable is the input, and another single variable is the output: $f: \mathbb{R} \mapsto \mathbb{R}$.

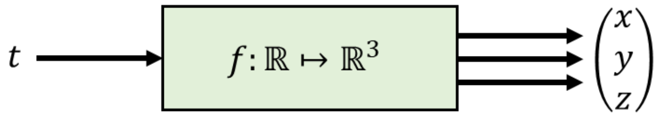
In our study of matrices, we considered functions that had N inputs and M outputs. However, we limited our study to **linear functions only**. This is like studying only linear functions $f(x) = ax$.

In the next lessons, we finally go deep into the most general case: a general function that has N inputs and M outputs. These are the functions that really surround us in the real world: the various fields (e.g. gravitational, electromagnetic), the varying density of objects, the flow of matter, etc. We will learn calculus (derivation, integration, maximization, etc.) in the general higher dimensional case.

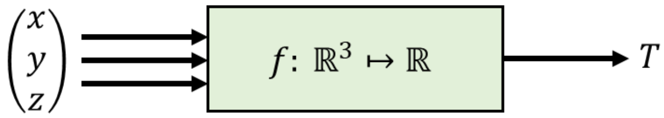
Single-variable scalar function (e.g. distance as function of time)



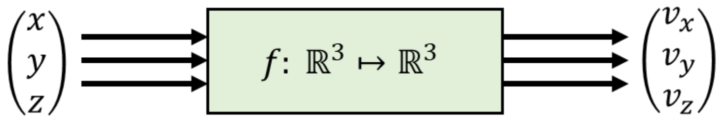
Single-variable vector function (e.g. position $\mathbf{r}$ as function of time)



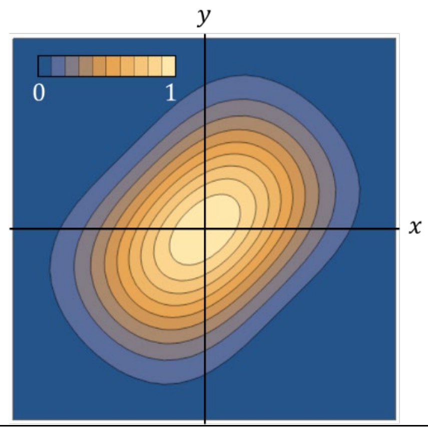
Multiple-variable scalar function (e.g. **scalar field**, e.g. temperature in a room)



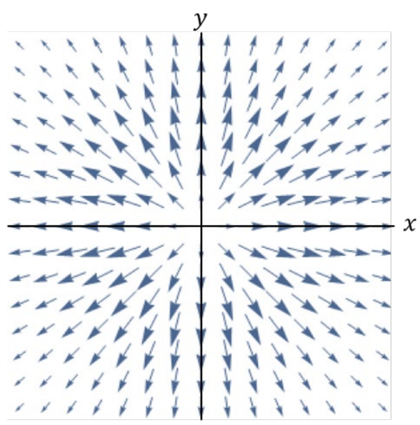
Multiple-variable vector function (e.g. **vector field**, e.g. air velocity in a room)



Scalar field: $T = e^{-x^2+y^2}(1 + \sin xy)$



Vector field: $\mathbf{v} = (x^2 + y^2 + 1)^{-5/2}(y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}})$



3.1 COORDINATE SYSTEMS: RECTANGULAR, POLAR, CYLINDRICAL, SPHERICAL

A. RECTANGULAR COORDINATES

When defining a field in space, the inputs of the function are called the coordinates; typically:

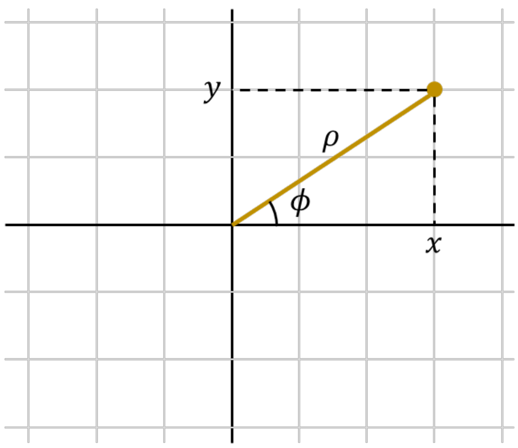
$$(x, y) \text{ or } (x, y, z)$$

These work great when we are working with straight/rectangular entities, but as soon as what we want to describe is curved, such as a circle, or radial, like the gravitational field of a point mass, the description becomes unnecessarily complicated and we resort to other systems of coordinates.

B. POLAR COORDINATES (2D)

In two dimensions, we can define points by using the radius ρ, and the azimuth ϕ

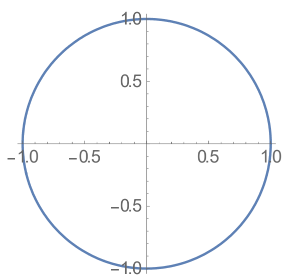
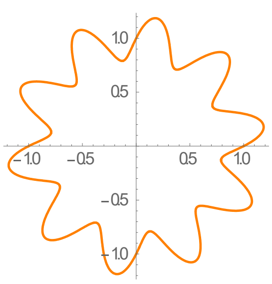
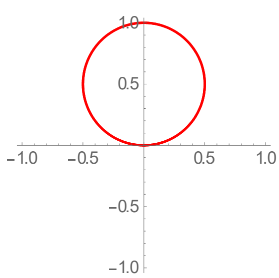
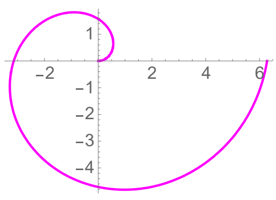
$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$



Conversion between (x, y) and (ρ, ϕ) is identical to conversion of complex numbers $z = x + iy = \rho e^{i\phi}$. As usual, be careful with $\tan^{-1}(y/x)$ to consider the correct quadrants. Computers usually have a function of two arguments called $\mathrm{atan2}(y, x)$.

1) **Problem**: Sketch the following curves in polar coordinates:

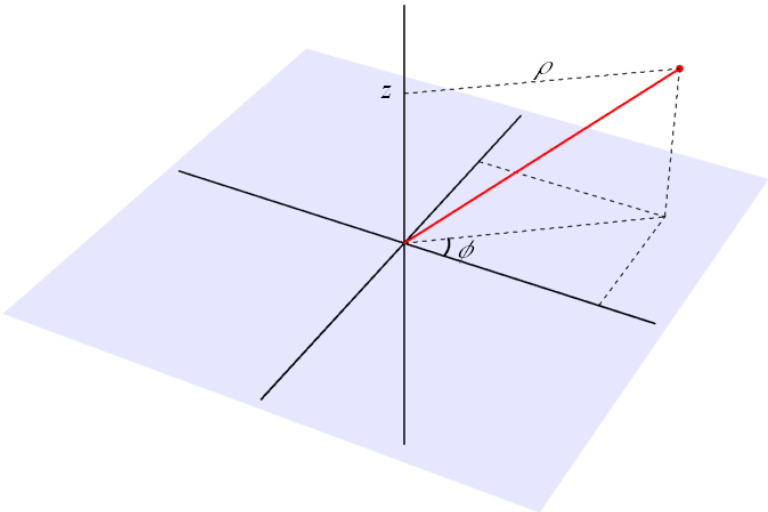
$\rho = 1$ $\rho = 1 + \frac{1}{5}\sin(10\phi)$ $\rho = \sin(\phi)$ $\rho = \phi$

C. CYLINDRICAL COORDINATES (3D)

Suitable for cylindrically symmetric objects: cylinders, tubes, etc. Identical to polar coordinates but adding the z-coordinate. Note the angle ϕ is still defined parallel to the xy plane.

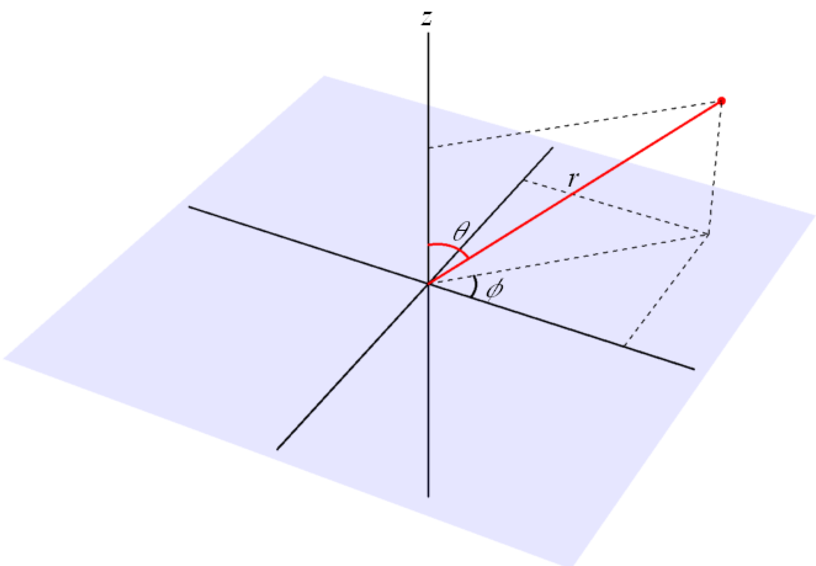
$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \quad \text{and} \quad \begin{cases} \rho = \sqrt{x^2+y^2} & \rho \in [0,\infty] \\ \phi = \tan^{-1}(y/x) & \phi \in [0,2\pi] \\ z = z \end{cases}$$



D. SPHERICAL COORDINATES (3D)

Suitable for spherically symmetric objects: spheres, onion-like structures, …

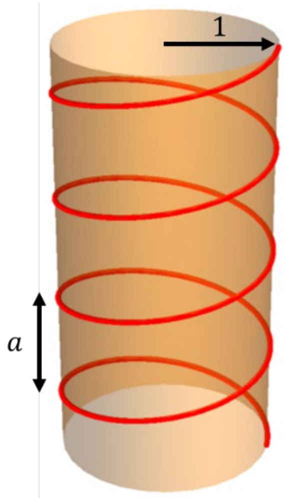
$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} & r \in [0,\infty] \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) & \theta \in [0,\pi] \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) & \phi \in [0,2\pi] \end{cases}$$



2) Describe/sketch the curves defined by the following equations in cylindrical and spherical coordinates

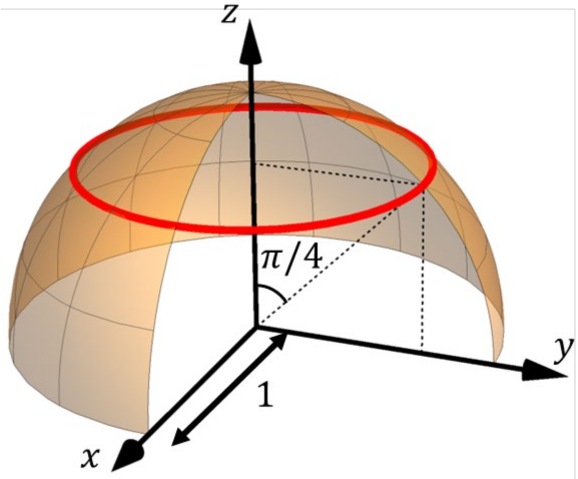
a) Cylindrical coordinates: $\rho = 1$; $\phi = \frac{2\pi}{a} z$ with $z \in [0, \infty]$

This describes a helix winded around a cylinder of radius 1. The pitch of the helix (height per turn) is equal to a, because ϕ completes a full revolution 0 to 2π when $z \in [0, a]$.



b) Spherical coordinates: $r = 1$; $\theta = \frac{\pi}{4}$; $\phi = t$ with $t \in [0, 2\pi]$

This describes a circle similar to a circle of constant latitude on earth. It is a circle, parallel to the XY plane, whose centre is at $\mathbf{c} = \cos\frac{\pi}{4}\,\hat{\mathbf{z}} = \frac{1}{\sqrt{2}}\,\hat{\mathbf{z}}$ and whose radius is equal to $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

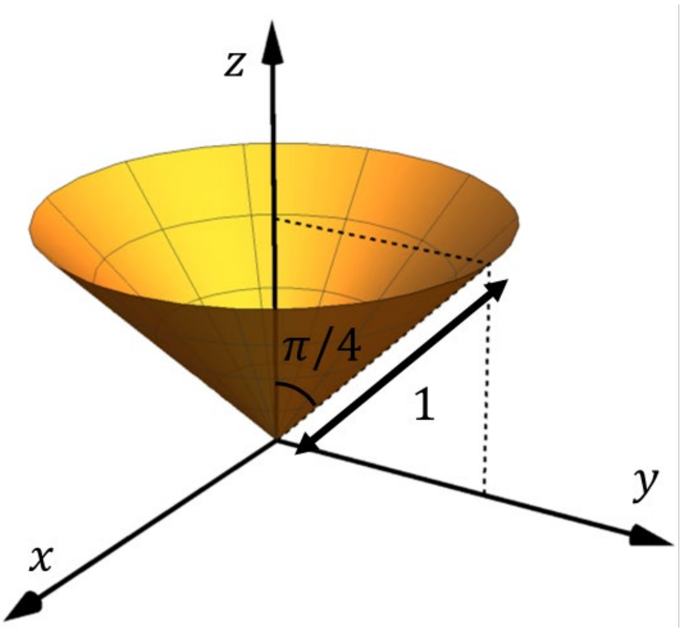


Notice that "1D curves" have a free parameter [e.g. $z \in [0, \infty]$ in (a) and $t \in [0, 2\pi]$ in (b)]. If we have two free parameters, then we can describe surfaces.

3) Describe/sketch the surfaces defined by the following equations in cylindrical and spherical coordinates:

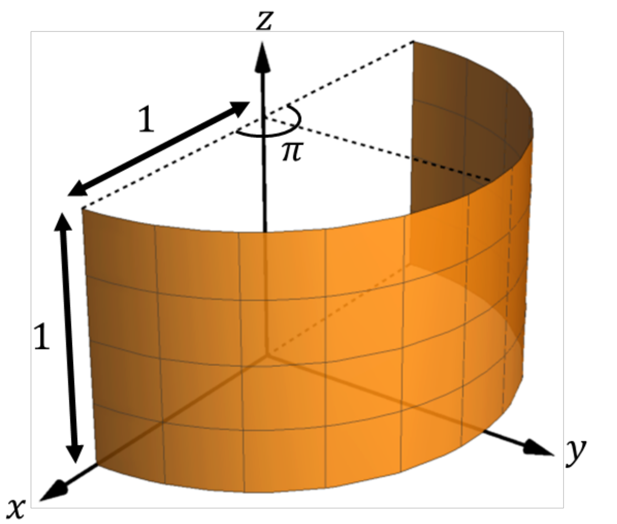
a) Spherical coordinates: $\theta = \frac{\pi}{4}$; $\phi \in [0,2\pi]$; $r \in [0,1]$

This describes the outer curved surface of an inverted cone with its tip at the origin, a tip angle of $\pi/2$, and a height of $1 \cos\frac{\pi}{4} = 1/\sqrt{2}$.



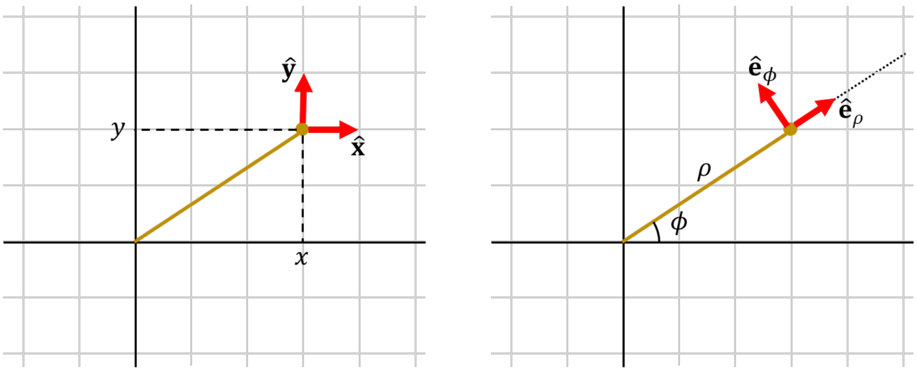
b) Cylindrical coordinates: $\rho = 1$; $\phi \in [0,\pi]$; $z \in [0,1]$

This describes the outer curved surface of a cylinder of radius 1 and height 1 which is "cut in half". i.e. keeping only the part in $y > 0$.



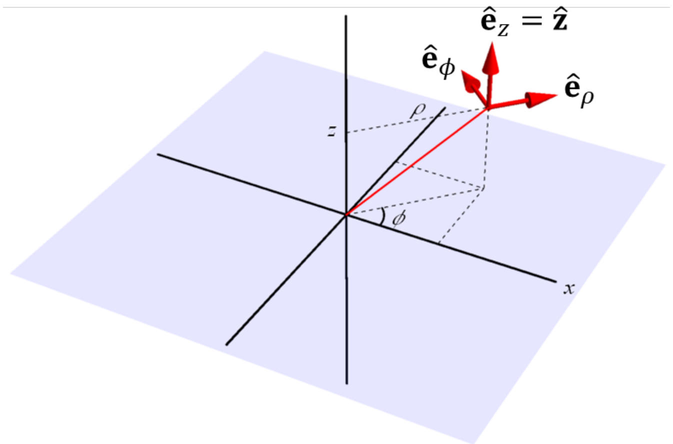
E. UNIT VECTORS IN POLAR, CYLINDRICAL, SPHERICAL COORDINATES

UNIT VECTORS FOR POLAR (2D) COORDINATES



$$\begin{cases} \hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

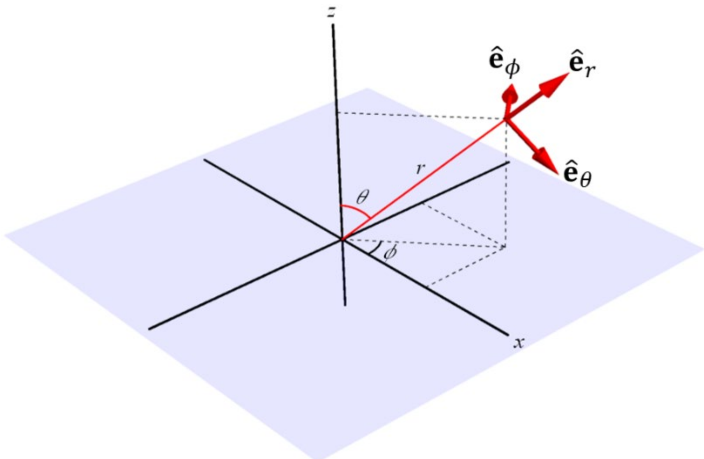
UNIT VECTORS FOR CYLINDRICAL (3D) COORDINATES



Same as for polar, but with the z direction unit vector:

$$\begin{cases} \hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{e}}_z = \hat{\mathbf{z}} \end{cases}$$

UNIT VECTORS FOR SPHERICAL (3D) COORDINATES

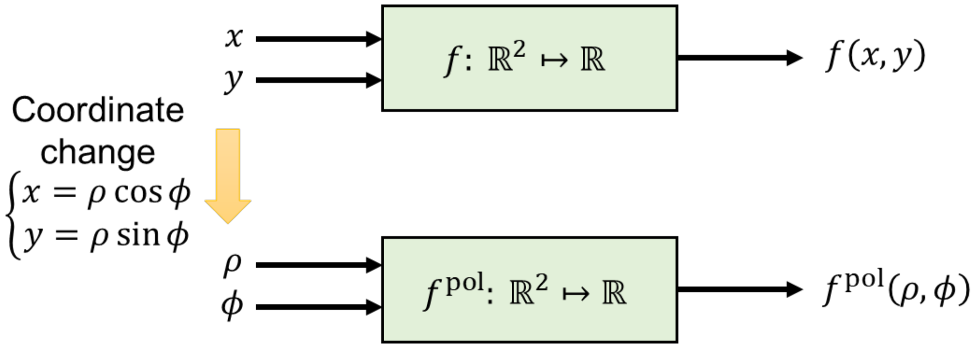


$$\begin{cases} \hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

F. CHANGING COORDINATES OF A FIELD (INPUT)

We use **coordinates** to define the **input values for our "field"**. A function $f(x, y, z)$ can be written in any other coordinates by simply doing the corresponding change of variables in the function. For example, converting a 2D function from rectangular to polar coordinates:

$$f(x, y) \quad \rightarrow \quad f^{\text{pol}}(\rho, \phi) = f(\rho\cos\phi\,, \rho\sin\phi)$$



Example: Convert $f(x, y) = \frac{e^{-x^2-y^2}}{\sqrt{x^2+y^2}}$ from rectangular to polar coordinates:

$$f^{\text{pol}}(\rho, \phi) = f(\rho\cos\phi\,, \rho\sin\phi) = \frac{e^{-\rho^2\cos^2\phi-\rho^2\sin^2\phi}}{\sqrt{\rho^2\cos^2\phi+\rho^2\sin^2\phi}} = \frac{e^{-\rho^2(\cos^2\phi+\sin^2\phi)}}{\sqrt{\rho^2(\cos^2\phi+\sin^2\phi)}} = \frac{e^{-\rho^2}}{\rho}.$$

The two functions f^{pol} and f **look different** in terms of the operations they do on their input variables, but they represent the **same scalar field**. This reminds us of how the same linear transformation could be represented by different matrices. **Normally we call them by the same name and use the input arguments as context** to know which one we are referring to: $f(x, y)$ or $f(\rho, \phi)$.

NOTE: Changing coordinates in NOT changing the BASIS of the input vector

I would like to clarify that changing the coordinates of the input position vector cannot be described as changing the basis of the input vector, in the linear algebra sense. This might seem confusing but is very simple. It is true that we can write the input position vector in various coordinates:

$\mathbf{r} = (x, y) = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$ in rectangular coordinates,

$\mathbf{r} = \rho\cos\phi\,\hat{\mathbf{x}} + \rho\sin\phi\,\hat{\mathbf{y}}$ in polar coordinates.

However, in both cases we are using the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ basis. We **cannot** interpret the input position vector (ρ, ϕ) as a vector with a basis $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi\}$ in the usual linear algebra sense:

$$\mathbf{r} = (\rho, \phi) = \rho\,\hat{\mathbf{e}}_\rho + \phi\,\hat{\mathbf{e}}_\phi. \quad [\textbf{WRONG}]$$

This is wrong, the dimensions don't even make sense! Why is this wrong? the basis vectors $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$ which represent unit vectors in the direction of ρ and ϕ, **are not well defined**! While $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ are always constant, $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$ depend on the position. Therefore, they cannot be used as a basis to describe the position vector itself. For example, what would $\mathbf{r} = 3\hat{\mathbf{e}}_\rho$ mean? Which direction?

However, the vectors $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$ **can** be defined **IF** the position $\mathbf{r}$ is given, therefore the basis **can be used to express the output vector**. Because the output vector is given at every specific position.

G. CHANGING BASIS OF A VECTOR FIELD (OUTPUT VECTOR)

Changing coordinates represents a change in the input of a field. However, we can also change the basis for the **output vector** (when the output is a vector in space, such as is the case in a vector field) into polar, cylindrical or spherical basis.

The interpretation is that we have a vector field $\mathbf{r} \to \mathbf{F}(\mathbf{r})$ and for **each** position $\mathbf{r}$, we **can** define a basis $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$ in which to expand the output vector $\mathbf{F}$. Therefore, the output of the function, the vector $\mathbf{F}(\mathbf{r})$ can be expressed in $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$, or in $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$.

Choice of coordinates (input) — 2D vector field — Choice of basis (output)

$$\mathbf{r} = \begin{cases} x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} \\ \rho\cos\phi\;\hat{\mathbf{x}} + \rho\sin\phi\;\hat{\mathbf{y}} \end{cases} \longrightarrow \boxed{\mathbb{R}^2 \mapsto \mathbb{R}^2} \longrightarrow \mathbf{F}(\mathbf{r}) = \begin{cases} F_x(\mathbf{r})\,\hat{\mathbf{x}} + F_y(\mathbf{r})\hat{\mathbf{y}} \\ F_\rho(\mathbf{r})\hat{\mathbf{e}}_\rho + F_\phi(\mathbf{r})\hat{\mathbf{e}}_\phi \end{cases}$$

> Do not confuse transforming the **coordinates** (the **input position vector** $\mathbf{r}$ of the function) with transforming the **field basis** (transforming the vector output **field itself** $\mathbf{F}(\mathbf{r})$).

To change the output basis, we can simply substitute in the expressions for basis vectors in terms of the other basis, or we can use the techniques we know from linear algebra (the change of basis is a linear operation and can be done with a change-of-basis matrix).

Example: Changing basis of the field from rectangular into polar coordinates

First obtain the change of basis-matrix $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$ to $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$

Remember we use the vectors $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$ as the columns: $\mathbf{A} = \begin{pmatrix} | & | \\ \hat{\mathbf{e}}_\rho & \hat{\mathbf{e}}_\phi \\ | & | \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

Since the columns are orthonormal, this is an orthogonal matrix, and therefore $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$, so no need to calculate the inverse. Therefore:

$$\begin{pmatrix} F_\rho \\ F_\phi \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \mathbf{A}^{\mathrm{T}}\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} F_x\cos\phi + F_y\sin\phi \\ -F_x\sin\phi + F_y\cos\phi \end{pmatrix}$$

Which gives us the components $\begin{pmatrix} F_\rho \\ F_\phi \end{pmatrix}$ in the new basis, as a function of the components $\begin{pmatrix} F_x \\ F_y \end{pmatrix}$.

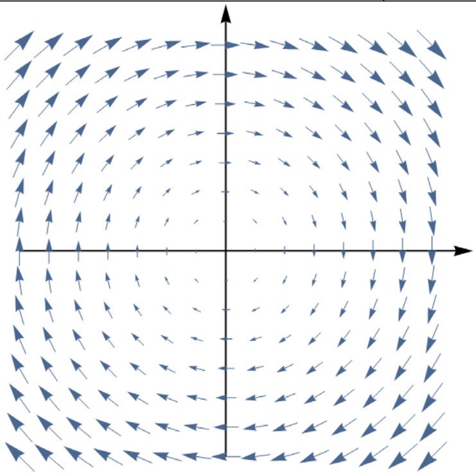
Remember that the vector components F_x and F_y are themselves functions of the position, which can be expressed in rectangular $F_x = F_x(x, y)$ or polar $F_x = F_x(\rho, \phi)$ coordinates. The change of basis matrix does not care which.

The same arguments can be applied to 3D fields, to convert into cylindrical or spherical coordinates and bases.

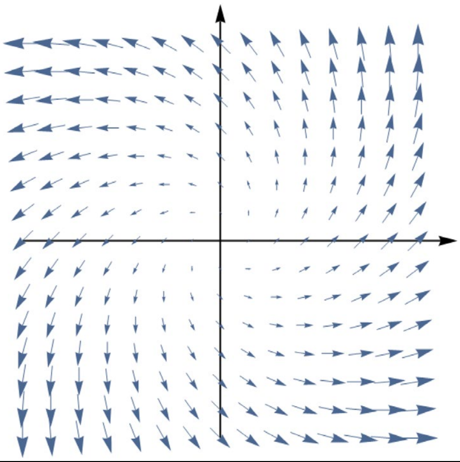
Examples:

To clarify the difference, consider how a given 2D vector field $\mathbf{F}(\mathbf{r})$ can be written in rectangular or polar coordinates and, for each of those two cases, the vector $\mathbf{F}$ can be expressed in rectangular or polar basis, having a total of 4 possible representations.

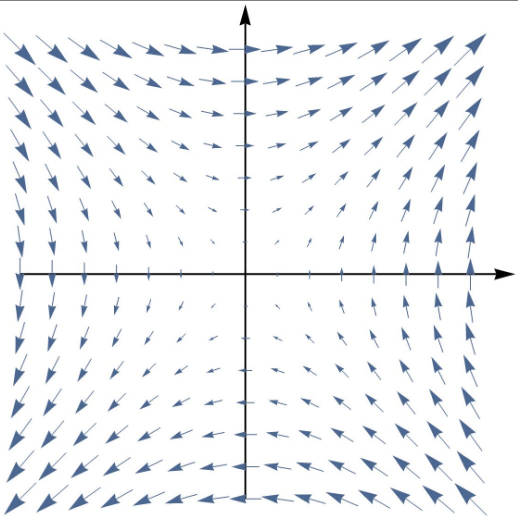
$\mathbf{F}(\mathbf{r})$		**Coordinates [input]**	
		Rectangular coordinates (x, y)	Polar coordinates (ρ, ϕ)
Basis [output]	Rectangular $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$	$\mathbf{F}(x, y) = y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}}$	$\mathbf{F}(\rho, \phi) = \rho \sin\phi\ \hat{\mathbf{x}} - \rho\cos\phi\ \hat{\mathbf{y}}$
	Polar $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$	$\mathbf{F}(x, y) = -\sqrt{x^2 + y^2}\,\hat{\mathbf{e}}_\theta$	$\mathbf{F}(\rho, \phi) = -\rho\,\hat{\mathbf{e}}_\phi$



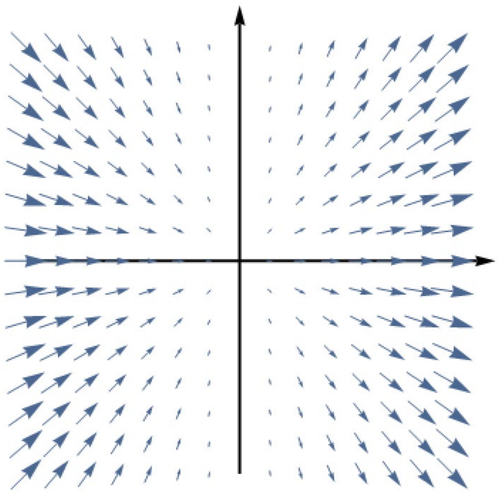
$\mathbf{F}(\mathbf{r})$		**Coordinates [input]**	
		Rectangular coordinates (x, y)	Polar coordinates (ρ, ϕ)
Basis [output]	Rectangular $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$	$\mathbf{F}(x, y) = (x - y)\,\hat{\mathbf{x}} + (x + y)\,\hat{\mathbf{y}}$	$\mathbf{F}(\rho, \phi) = \rho(\cos\phi - \sin\phi)\,\hat{\mathbf{x}}$ $+\rho(\cos\phi + \sin\phi)\hat{\mathbf{y}}$
	Polar $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$	$\mathbf{F}(x, y) = \sqrt{x^2 + y^2}\,\hat{\mathbf{e}}_\rho +$ $\sqrt{x^2 + y^2}\,\hat{\mathbf{e}}_\phi$	$\mathbf{F}(\rho, \phi) = \rho\big(\hat{\mathbf{e}}_\rho + \hat{\mathbf{e}}_\phi\big)$



$\mathbf{F}(\mathbf{r})$		**Coordinates [input]**	
		Rectangular coordinates (x, y)	Polar coordinates (ρ, ϕ)
Basis [output]	Rectangular $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$	$\mathbf{F}(x, y) = y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$	$\mathbf{F}(\rho, \phi) = \rho \sin\phi \;\hat{\mathbf{x}} + \rho\cos\phi \;\hat{\mathbf{y}}$
	Polar $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$	$\mathbf{F}(x, y) = \dfrac{2xy}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_\rho + \dfrac{x^2-y^2}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_\phi$	$\mathbf{F}(\rho, \phi) = \rho\sin(2\phi)\,\hat{\mathbf{e}}_\rho + \rho\cos(2\phi)\,\hat{\mathbf{e}}_\phi$



$\mathbf{F}(\mathbf{r})$		**Coordinates [input]**	
		Rectangular coordinates (x, y)	Polar coordinates (ρ, ϕ)
Basis [output]	Rectangular $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$	$\mathbf{F}(x, y) = \frac{x^2}{\sqrt{x^2+y^2}}\;\hat{\mathbf{x}} + \frac{xy}{\sqrt{x^2+y^2}}\;\hat{\mathbf{y}}$	$\mathbf{F}(\rho, \phi) = \rho\cos^2\phi\;\hat{\mathbf{x}} + \rho\cos\phi\sin\phi\;\hat{\mathbf{y}}$
	Polar $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$	$\mathbf{F}(x, y) = x\,\hat{\mathbf{e}}_\rho$	$\mathbf{F}(\rho, \phi) = \rho\cos\phi\;\hat{\mathbf{e}}_\rho$



PROBLEMS:

4) **Problems**: Given any of the four versions of the fields in the examples above, calculate the other three versions (i.e. fill the table starting from only one element).

$\mathbf{F}(\mathbf{r})$		**Coordinates [input]**	
		Rectangular coordinates (x, y)	Polar coordinates (ρ, ϕ)
Basis [output]	Rectangular $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$	$\mathbf{F}(x, y) = y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}}$	(1)
	Polar $\{\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi\}$	(3)	(2)

1. **Convert to polar coordinates (keeping rectangular basis)**

Simply substitute $x = \rho\cos\phi$ and $y = \rho\sin\phi$

$$\mathbf{F}(x, y) \to \mathbf{F}(\rho\cos\phi\,, \rho\sin\phi) = \rho\sin\phi\;\hat{\mathbf{x}} - \rho\cos\phi\;\hat{\mathbf{y}} = \rho(\sin\phi\,\hat{\mathbf{x}} - \cos\phi\,\hat{\mathbf{y}})$$

2. **Convert this form into polar basis by using a change of basis matrix:**

$$\mathbf{A} = \begin{pmatrix} | & | \\ \hat{\mathbf{e}}_\rho & \hat{\mathbf{e}}_\phi \\ | & | \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$\begin{pmatrix} F_\rho \\ F_\phi \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \mathbf{A}^{\mathrm{T}}\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} F_x\cos\phi + F_y\sin\phi \\ -F_x\sin\phi + F_y\cos\phi \end{pmatrix}$$

$$F_\rho = F_x\cos\phi + F_y\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$
$$F_\phi = -F_x\sin\phi + F_y\cos\phi = -\rho\sin\phi\sin\phi - \rho\cos\phi\cos\phi = -\rho(\sin^2\phi + \cos^2\phi) = -\rho$$

So that: $\mathbf{F}(r, \theta) = 0\hat{\mathbf{e}}_\rho - \rho\hat{\mathbf{e}}_\phi$

3. **Convert the coordinates of (2) into rectangular coordinates**

Simply substitute $\rho = \sqrt{x^2 + y^2}$, so that:

$$\mathbf{F}(x, y) = 0\hat{\mathbf{e}}_\rho - \sqrt{x^2 + y^2}\,\hat{\mathbf{e}}_\phi$$

Note, **alternative path**: We could have first changed the basis, i.e. go directly to (3), by using the change of basis matrix, but expressed in rectangular coordinates, that is:

$$\mathbf{A} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \frac{1}{\sqrt{x^2 + y^2}}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

$$\begin{pmatrix} F_\rho \\ F_\phi \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \mathbf{A}^{\mathrm{T}}\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \frac{1}{\sqrt{x^2 + y^2}}\begin{pmatrix} F_x x + F_y y \\ -F_x y + F_y x \end{pmatrix}$$

$$F_\rho = \frac{1}{\sqrt{x^2 + y^2}}\left(F_x x + F_y y\right) = \frac{1}{\sqrt{x^2 + y^2}}(yx - xy) = 0$$
$$F_\phi = \frac{1}{\sqrt{x^2 + y^2}}\left(-F_x x + F_y y\right) = \frac{1}{\sqrt{x^2 + y^2}}(-y^2 - x^2) = -\sqrt{x^2 + y^2}$$

Try the rest of the examples for yourself

5) Problem: The gravitational field of a mass M placed at the origin is given by $\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{e}}_r$ in spherical coordinates and spherical basis. Complete a table as above, including rectangular and spherical coordinates and basis.

Solution:

1. Rectangular coordinates but spherical basis:

We just substitute the expression for the r coordinate and arrive at:

$$\mathbf{g} = -\frac{GM}{x^2 + y^2 + z^2}\hat{\mathbf{e}}_r$$

2. Rectangular basis but spherical coordinates:

a) Changing basis using a matrix:

The change-of-basis matrix for converting $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\}$ into $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ is given by a matrix whose columns are the vectors $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\}$ written themselves in $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ basis. For this we can look-up the spherical basis vectors:

$$\begin{cases} \hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

$$\mathbf{A} = \begin{pmatrix} | & | & | \\ \hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_\phi \\ | & | & | \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

Therefore, the vector:

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{e}}_r = -\frac{GM}{r^2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ in basis } \{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\}$$

gets converted into:

$$\mathbf{g} = -\frac{GM}{r^2}\begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\frac{GM}{r^2}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \text{ in basis } \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$$

b) Changing basis by substituting in the vectors directly:

Since we are given $\hat{\mathbf{e}}_r$ in a list above, we can simply substitute it:

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{e}}_r = -\frac{GM}{r^2}(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})$$

this second method was of course much simpler, in this case.

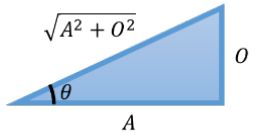
3. Rectangular basis and rectangular coordinates

Once the basis is rectangular (from 2) we now change the coordinates. For this, we "simply" substitute the expressions for r, θ, ϕ:

$$\mathbf{g}(r,\theta,\phi) \to \mathbf{g}(r(x,y,z),\theta(x,y,z),\phi(x,y,z)) = \mathbf{g}\left(\sqrt{x^2+y^2+z^2}, \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \tan^{-1}\left(\frac{y}{x}\right)\right)$$

This leads to a horrible expression. Fortunately, r only appears as r^2, which will remove the square root, and the angles only appear as their sines and cosines. The sines and cosines of inverse tangents can be simplified by drawing a suitable triangle:

$$\sin\left(\tan^{-1}\left(\frac{O}{A}\right)\right) = \frac{O}{\sqrt{O^2+A^2}}$$
$$\cos\left(\tan^{-1}\left(\frac{O}{A}\right)\right) = \frac{A}{\sqrt{O^2+A^2}}$$



Therefore:

$$\mathbf{g} = -\frac{GM}{r^2}(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})$$
$$= -\frac{GM}{x^2+y^2+z^2}\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{z}}\right)$$
$$= -\frac{GM}{(x^2+y^2+z^2)^{3/2}}(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})$$

which is in rectangular coordinates **and** basis.

Note there is a **fast way** to arrive at this result. We can realise that $\hat{\mathbf{e}}_r = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{\mathbf{r}}{r}$

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{e}}_r = -\frac{GM}{r^2}\frac{\mathbf{r}}{r} = -\frac{GM}{r^3}\mathbf{r}$$

and then substitute $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$ and $r = \sqrt{x^2+y^2+z^2}$, to arrive directly at the answer! This convenient shortcut only works with spherically symmetric fields.

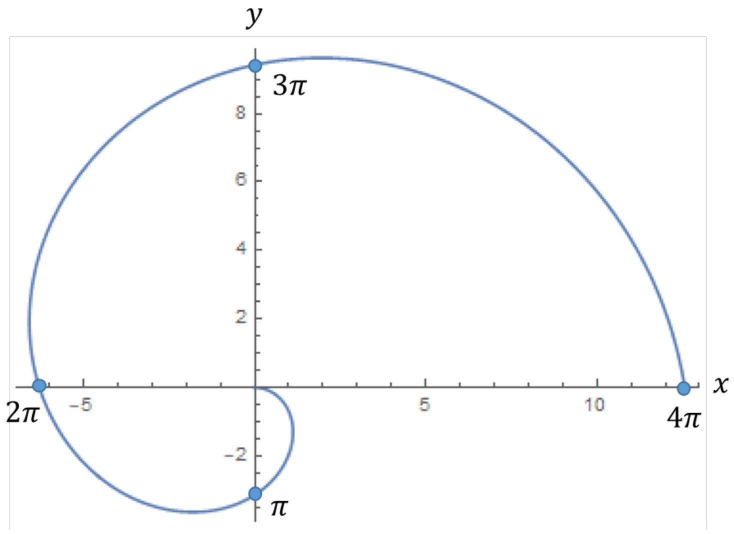
So, we have all the required expressions to complete the table:

Gravitational field: $\mathbf{g}(\mathbf{r})$		**Coordinates [input]**	
		Rectangular coordinates (x,y,z)	Spherical coordinates (r,θ,ϕ)
Basis [output]	Rectangular $\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$	$-GM\dfrac{x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}$	$-\dfrac{GM}{r^2}(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})$
	Spherical $\{\hat{\mathbf{e}}_r,\hat{\mathbf{e}}_\theta,\hat{\mathbf{e}}_\phi\}$	$-\dfrac{GM}{x^2+y^2+z^2}\hat{\mathbf{e}}_r$	$-\dfrac{GM}{r^2}\hat{\mathbf{e}}_r$

Note that we are in three dimensions, so in this table we could include a third row and column, for cylindrical coordinates and basis.

6) Sketch the following curve in polar coordinates $\rho = 4\pi - 2\phi$ for $\phi \in [0,2\pi]$, and indicate the values of all intercepts with the x and y axes.

By substituting some values: $\phi = [0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi]$ results in $\rho = [4\pi, 3\pi, 2\pi, \pi, 0]$



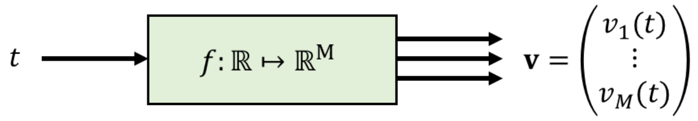
3.2 INTEGRATING AND DIFFERENTIATING VECTORS

Your knowledge of calculus in the past has involved functions $f(x)$, with one input and one output output $f: \mathbb{R} \mapsto \mathbb{R}$. We are going to learn how to work with functions which have N variables (inputs) and have M components (outputs), that is: $f: \mathbb{R}^N \mapsto \mathbb{R}^M$.

We can look at two parts of this problem: (a) how to deal with multiple outputs, and (b) how to deal with multiple inputs. In fact, (a) is easy. So, let's first get the trivial part out of the way: integration and differentiation of vectors.

A. INTEGRATION OF VECTORS

Consider functions with only one input but multiple outputs $f: \mathbb{R} \mapsto \mathbb{R}^M$. This is equivalent to **a vector which depends on one single variable**.



For example: the position of an object as a function of time; the wind velocity (3D vector) at a weather vane as a function of time; …

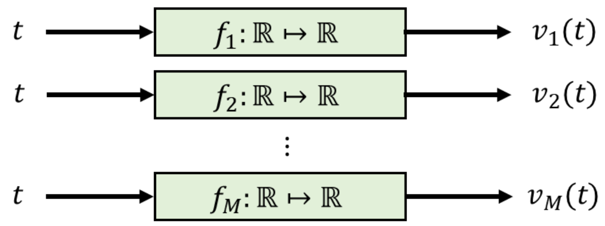
So, how do we integrate a vector?

$$\mathbf{I} = \int_a^b \mathbf{v}(t)\, \mathrm{d}t$$

The solution is simple and can be justified **mathematically** by applying the linearity of the integrals (when the basis vectors are constant):

$$\int_a^b \mathbf{v}(t)\, \mathrm{d}t = \int_a^b (v_1(t)\mathbf{e}_1 + \cdots v_M(t)\mathbf{e}_M\,)\mathrm{d}t = \left(\int v_1(t)\, \mathrm{d}t\right)\mathbf{e}_1 + \cdots + \left(\int v_M(t)\, \mathrm{d}t\right)\mathbf{e}_M$$

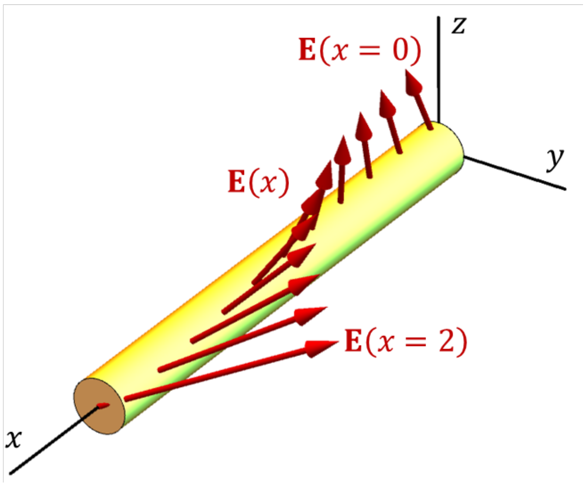
If you think about it, this vector function of a single variable $\mathbf{v}(t)$ is equivalent to defining M different functions, one for each component of the vector:



So, indeed, the most natural solution is to integrate each component separately:

$$\mathbf{I} = \int_a^b \mathbf{v}(t)\, \mathrm{d}t = \int_a^b \begin{pmatrix} v_1(t) \\ \vdots \\ v_M(t) \end{pmatrix} \mathrm{d}t = \begin{pmatrix} \int v_1(t)\, \mathrm{d}t \\ \vdots \\ \int v_M(t)\, \mathrm{d}t \end{pmatrix}$$

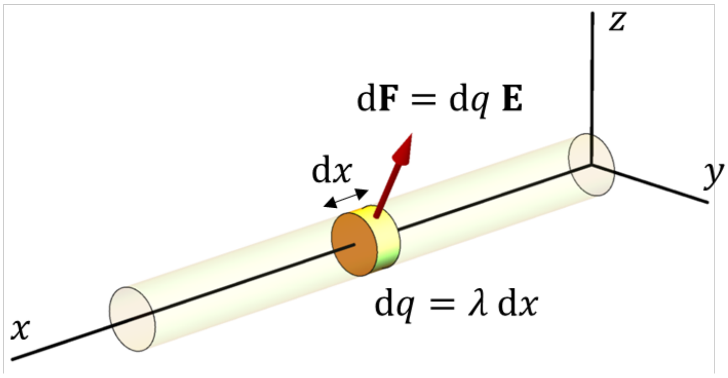
1) Problem: A charged straight metal rod (from $x = 0$ to $x = 2$) has a constant charge linear density of λ C/m and is placed in a region with an external electric field $\mathbf{E}(x) = (1-x)\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$ V/m, shown below. Calculate the **net electric force** acting on the rod.



Solution: We want to obtain the force, which is a vector, therefore this will clearly involve an integral of a vector. The integrand must be a differential of force $\mathbf{dF}$ so that the total force.

$$\mathbf{F} = \int \mathbf{dF}$$

Always think about what you are doing. We need to find the differential of the force $\mathbf{dF}$, as a function of position. We expect several things from it: (1) The differential of the force $\mathbf{dF}$ must be a vector, so that its integral is a vector; (2) The differential of the force will hopefully be written in terms of $\mathrm{d}x$, so that we can integrate along x; (3) What is the meaning of the differential of the force? It must be the force acting on a differential length $\mathrm{d}x$ of the rod. (4) The force acting on a differential length $\mathrm{d}x$ is equivalent to the force acting on a differential charge $\mathrm{d}q$.



The equation of the electric force is $\mathbf{F} = q\mathbf{E}$ therefore, for a differential charge $\mathrm{d}q$, the force will be $\mathbf{dF} = \mathrm{d}q\ \mathbf{E}$. The differential charge is the charge contained on a differential length $\mathrm{d}x$ of the rod, which by definition of the charge density is $\mathrm{d}q = \lambda\ \mathrm{d}x$.

Putting it all together: $d\mathbf{F}(x) = \mathrm{d}q\ \mathbf{E}(x) = \lambda\ \mathrm{d}x\ \mathbf{E}(x) = \lambda[(1-x)\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + 2\hat{\mathbf{z}}]\mathrm{d}x$.

So we substitute this into our integral and solve it:

$$\mathbf{F} = \int \mathbf{dF} = \int_0^2 \lambda[(1-x)\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + 2\hat{\mathbf{z}}]\mathrm{d}x = \begin{pmatrix}\text{integrate}\\ \text{each component}\end{pmatrix} = 0\hat{\mathbf{x}} + \frac{8\lambda}{3}\hat{\mathbf{y}} + 4\lambda\hat{\mathbf{z}}$$

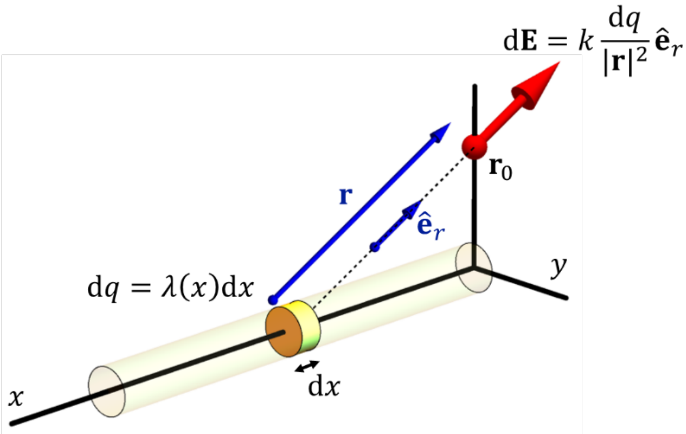
2) **Problem**: A charged metal rod spans the line segment $(0,0,0)$ to $(2,0,0)$. The rod has a varying linear charge density of $\lambda(x) = \lambda_0(1 + x^2)^{3/2}$ C/m. Calculate the **electric field** created by the rod at the point $\mathbf{r_0} = (0,0,1)$.

Solution: As before, we can start by realising that we will have to integrate $\mathrm{d}\mathbf{E}$ to calculate the total electric field $\mathbf{E}$:

$$\mathbf{E} = \int \mathrm{d}\mathbf{E}$$

And now think about what $\mathrm{d}\mathbf{E}$ must be. It must be the electric field created at r_0 by a differential of charge $\mathrm{d}q = \lambda\, \mathrm{d}x$ along the rod. The electric field $\mathbf{E}$ created at $\mathbf{r}_0$ by a charge q is equal to $\mathbf{E} = \hat{\mathbf{e}}_r kq/|\mathbf{r}|^2$, where $\mathbf{r}$ is the vector pointing from the location of q to the location $\mathbf{r}_0$ where we are calculating the electric field, and $\hat{\mathbf{e}}_r = \mathbf{r}/|\mathbf{r}|$ is the unit vector in that same direction.

Therefore, we can apply that equation for each $\mathrm{d}q$.



Just remember that almost everything in this expression is a function of x:

$$d\mathbf{E} = k\frac{\mathrm{d}q(x)}{|\mathbf{r}(x)|^2}\hat{\mathbf{e}}_r(x) = k\frac{\mathrm{d}x\,\lambda(x)}{|\mathbf{r}(x)|^2}\hat{\mathbf{e}}_r(x)$$

Also note that we can simplify:

$$\frac{1}{|\mathbf{r}|^2}\hat{\mathbf{e}}_r = \frac{1}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$$

Now, the position vector $\mathbf{r}$ which points from the position of the charge $\mathrm{d}q$, which is $x\,\hat{\mathbf{x}}$, to the position $\mathbf{r}_0$ can be written as: $\mathbf{r}(x) = \mathbf{r}_0 - x\,\hat{\mathbf{x}} = (0,0,1) - (x,0,0) = (-x)\hat{\mathbf{x}} + \hat{\mathbf{z}}$. Therefore, $|\mathbf{r}(x)| = (x^2 + 1^2)^{1/2}$, and putting it all together:

$$d\mathbf{E} = \mathrm{d}x\; k\;\lambda(x)\frac{\mathbf{r}(x)}{|\mathbf{r}(x)|^3} = \mathrm{d}x\; k\;\lambda(x)\frac{(-x)\hat{\mathbf{x}} + \hat{\mathbf{z}}}{(1 + x^2)^{3/2}}$$

Which we can substitute into the integral. Also, use $\lambda(x) = \lambda_0(1 + x^2)^{3/2}$ from the given data:

$$\mathbf{E} = \int \mathrm{d}\mathbf{E} = \int_0^2 \mathrm{d}x\; k\;\lambda_0(1 + x^2)^{3/2}\frac{(-x)\hat{\mathbf{x}} + \hat{\mathbf{z}}}{(1 + x^2)^{3/2}} = k\lambda_0\int_0^2 \mathrm{d}x\;[(-x)\hat{\mathbf{x}} + \hat{\mathbf{z}}] = k\lambda_0(-2\hat{\mathbf{x}} + 2\hat{\mathbf{z}})$$

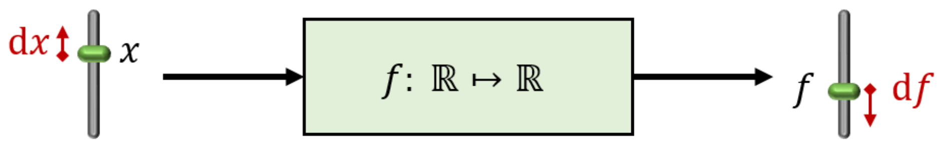
B. DIFFERENTIATION OF VECTORS

Remember: The derivative of a function is telling us how much the output (f) changes when we change the input (x), i.e. it tells us the ratio $\mathrm{d}f/\mathrm{d}x$ such that:

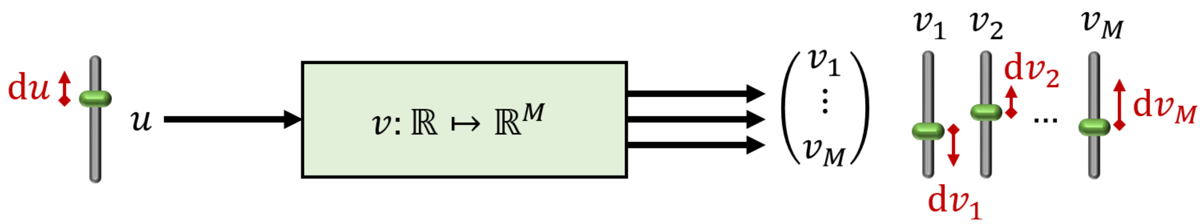
$$x \mapsto f$$

$$x + \mathrm{d}x \mapsto f + \mathrm{d}f$$

You can picture a function as a mechanism that connects a moving knob in the input with a moving knob in the output. The derivative is the ratio of how much the output knob moves when you move the input. The ratio tends to the derivative when the nudge is so small that the relation is linear:



But what if we have multiple outputs? Then we have a vector which depends on a single variable $\mathbf{v}(u)$. You can still picture it with knobs. Changing the input changes all the outputs:



How do we do a differentiation in this context? The answer is obvious, just calculate the change on each output separately, that is, do the derivative of each component.

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}u} = \begin{pmatrix} \mathrm{d}v_1/\mathrm{d}u \\ \mathrm{d}v_2/\mathrm{d}u \\ \vdots \\ \mathrm{d}v_M/\mathrm{d}u \end{pmatrix}$$

This intuitive picture can be justified mathematically by applying the linearity of the differentiation:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}u} = \frac{\mathrm{d}}{\mathrm{d}u}(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_M\mathbf{e}_M) = \frac{\mathrm{d}v_1}{\mathrm{d}u}\mathbf{e}_1 + \frac{\mathrm{d}v_2}{\mathrm{d}u}\mathbf{e}_2 + \cdots + \frac{\mathrm{d}v_M}{\mathrm{d}u}\mathbf{e}_M$$

Also, we can derive it formally from the definition of a derivative:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}u} = \lim_{\Delta u \to 0} \frac{\mathbf{v}(u + \Delta u) - \mathbf{v}(u)}{\Delta u}$$

In fact, this definition gives us some intuition. The derivative of a vector is another vector which tells us the rate of change of the vector **in both amplitude and direction**.

This simple calculation has very useful physical interpretations, especially when the vector output represents a position vector in space.

- When the input variable is time and the output is the position vector $\mathbf{r}(t)$, the successive derivatives tell us about velocity $\mathbf{v}(t) = \mathrm{d}\mathbf{r}(t)/\mathrm{d}t$, acceleration $\mathbf{a}(t) = \mathrm{d}^2\mathbf{r}(t)/\mathrm{d}t^2$, etc.

3) Example: A particle follows the following path as a function of time:

$$\mathbf{r}(t) = \begin{pmatrix} \frac{1}{4}\sin(3\pi t) + \frac{t}{10} \\ \frac{1}{2} - t \\ t\cos(t) \end{pmatrix}$$

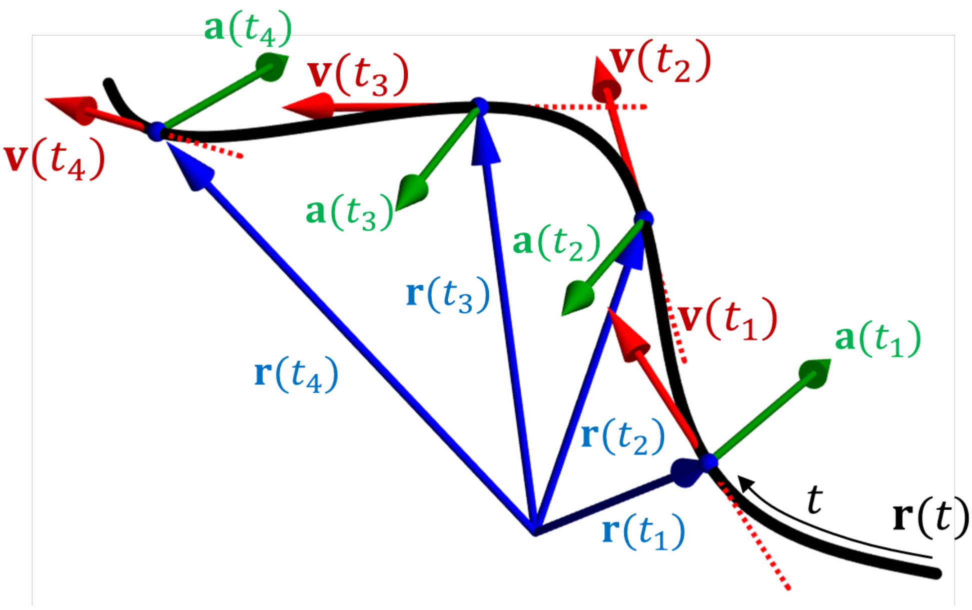
Calculate the instantaneous velocity and acceleration.

Solution: This is a $\mathbb{R} \to \mathbb{R}^3$ function, so the time derivative just needs to be applied to each component.

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \\ \mathrm{d}z/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} \frac{3\pi}{4}\cos(3\pi t) + \frac{1}{10} \\ -1 \\ \cos(t) - t\sin(t) \end{pmatrix}$$

$$\mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}v_x/\mathrm{d}t \\ \mathrm{d}v_y/\mathrm{d}t \\ \mathrm{d}v_z/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} -\frac{9\pi^2}{4}\sin(3\pi t) \\ 0 \\ -2\sin(t) - t\cos(t) \end{pmatrix}$$

If we actually plot this path (by joining the ends of the position vectors $\mathbf{r}$ placed at the origin), and plot also the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors placed at the appropriate position, then we clearly see that the velocity is a vector always tangent to the curve, and the acceleration is always pointing in the direction in which the curve is being bent. This is a section of the curve for times $t \in [0,1]$.



TANGENT VECTOR

In general, when $\mathbf{r}(u)$ defines a curve in space, the derivative $\mathrm{d}\mathbf{r}/\mathrm{d}u$ gives us a **tangent vector** to the curve.

DIFFERENTIAL OF A VECTOR

We can define the differential of a vector $\mathrm{d}\mathbf{r}$ in a similar way to that of a scalar. Consider a change $\Delta\mathbf{r}$ in a vector caused by a change Δt in its parameter. When $\Delta t \to dt$ becomes increasingly small (approaching zero, becoming a differential) then $\Delta\mathbf{r} \to \mathrm{d}\mathbf{r}$.

The differential of a vector is itself a vector, i.e. it has components in a basis. For example, we can write the differential of position as:

$$\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\mathrm{d}t = \mathbf{v}\,\mathrm{d}t$$

So that $\mathrm{d}\mathbf{r}$ is a vector that has a magnitude $|\mathbf{v}|\,\mathrm{d}t$ and is oriented in the same direction as $\mathbf{v}$.

DIFFERENTIATING SCALAR AND CROSS PRODUCTS

The rule for differentiating dot and cross products take identical form to the product rule with scalars:

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathbf{a}\cdot\mathbf{b}) = \left(\mathbf{a}\cdot\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}u}\right) + \left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}u}\cdot\mathbf{b}\right)$$

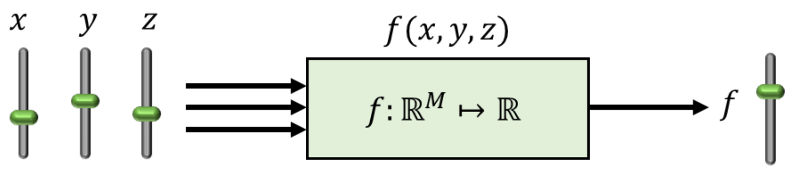
$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathbf{a}\times\mathbf{b}) = \left(\mathbf{a}\times\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}u}\right) + \left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}u}\times\mathbf{b}\right)$$

3.3 PARTIAL DIFFERENTIATION

We have seen that integration and differentiation is easy to do on a function with multiple outputs. The interesting cases appear when a function has **multiple inputs**. Let's start simple, with the case of **multiple inputs and single output**.

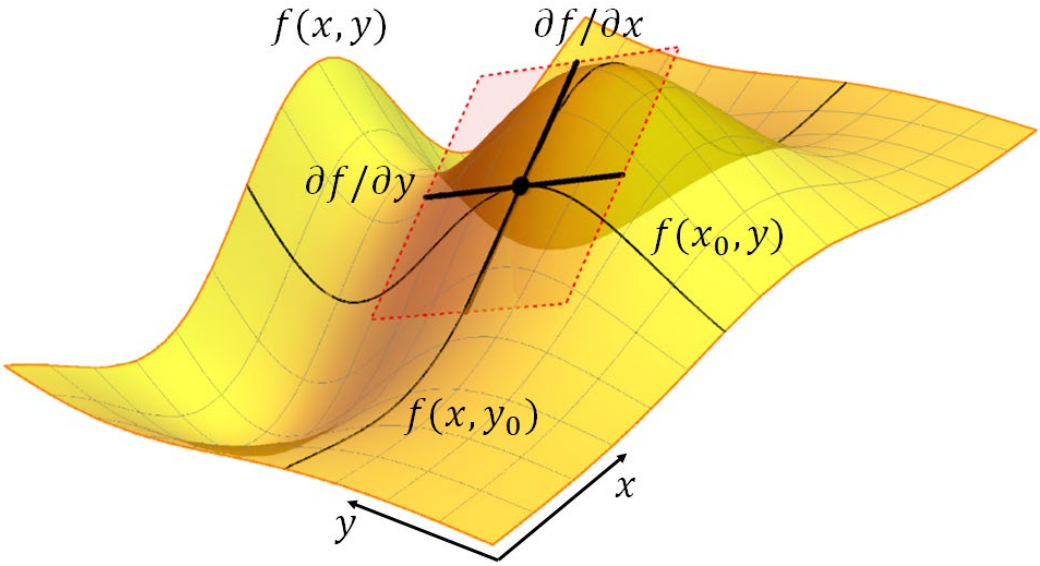
A. MULTIPLE INPUTS: PARTIAL DERIVATIVES

A function of multiple inputs can be pictured as follows:



Each of the input knobs will affect the output. The **partial derivative** tells us how much f moves when we move each of the input knobs, **while keeping the others constant**. For example, we look at the ratio $\Delta f/\Delta x$ when Δx goes to zero. Of course, this ratio depends on the value of **all** the inputs.

Let's consider for simplicity functions of two variables $f(x,y)$. These can be represented as the variation in height with position in a mountainous landscape



It is clear that $f(x,y)$ will have a gradient in all directions in the xy-plane. However, we can consider the simpler case of finding the rate of change of $f(x,y)$ in the positive x- and y- directions. These are the partial derivatives with respect to x and y respectively.

We may define the partial derivative with respect to x by defining a one-variable function of x when y is held fixed and treated as a constant: $f(x,y_0)$. To signify that the derivative is with respect to x, but at the same time to recognize that a derivative with respect to y also exists, we denote the derivative using the partial derivative sign as: $\partial f/\partial x$. It is formally defined as the limit:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

The definition is trivially extended to the case of M input variables by keeping all the others constant. Sometimes when teaching about partial derivatives, it is useful and common practice to indicate those variables that are held constant by writing them as subscripts to the derivative symbol. Thus, for the two variable function $f(x,y)$ we can write:

$$\frac{\partial f}{\partial x} \equiv \left(\frac{\partial f}{\partial x}\right)_y$$

Importantly, the result of this partial derivative is another function which depends on the same inputs. It is common to denote the partial derivative with subscripts of the function.

$$\frac{\partial f}{\partial x} = f_x(x,y)$$

$$\frac{\partial f}{\partial y} = f_y(x,y)$$

HIGHER ORDER PARTIAL DERIVATIVES:

Now we can apply additional partial derivatives to these new functions. These are partial derivatives of higher order. Interestingly, we may change the variable held constant in each successive differentiation, leading to cross partial derivatives. For example, all possible second order partial derivatives of a 2-variable function $f(x,y)$ are:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

Only three of these are independent, because it can be shown that provided the second partial derivatives are continuous at the point in question, then the following relation is always obeyed:

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

You may ask **"Why?"**. See formal proof: https://math.stackexchange.com/questions/965018/why-does-the-order-not-matter-partial-d

"Ok, I believe the proof… but still… **WHY?"** See different geometrical interpretations:
https://math.stackexchange.com/questions/942538/geometric-interpretation-of-mixed-partial-derivatives

1) Find all first and second partial derivatives of the function $f(x,y) = 2x^3y^2 + y^3$

The first partial derivatives are calculated by assuming the other variable is a constant:

$$\frac{\partial f}{\partial x} = 6x^2y^2 \qquad \frac{\partial f}{\partial y} = 4x^3y + 3y^2$$

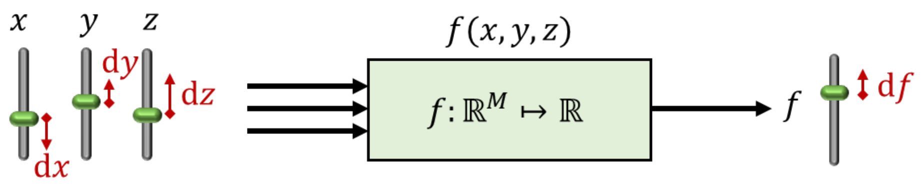
And the second partial derivatives are obtained by performing partial derivatives on the first ones:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = 12xy^2 \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = 4x^3 + 6y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = 12x^2y \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = 12x^2y$$

B. TOTAL DIFFERENTIAL

What if we move all the inputs simultaneously? What happens then with the output?



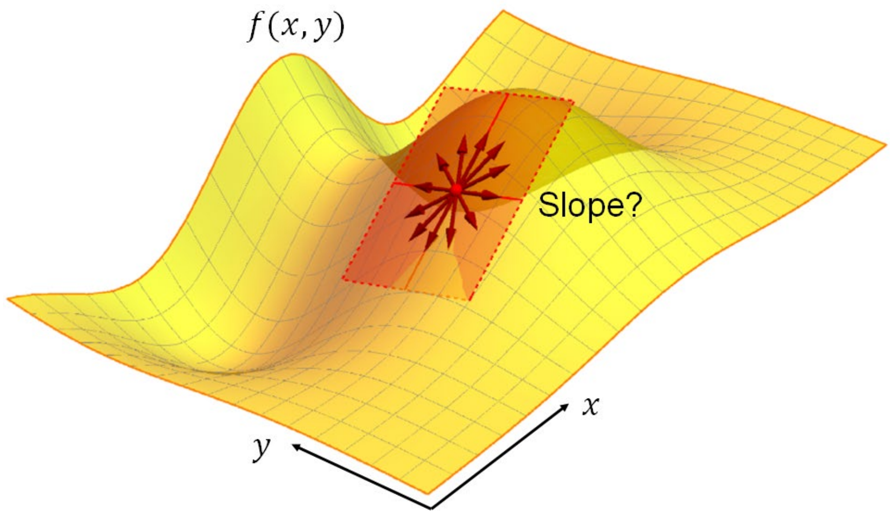
Clearly, the movement of the output $\mathrm{d}f$ must depend on the movement of each of the inputs $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$. In fact, we know how much each input changes the output, so if the changes are tiny, the total change in f will be the sum of the changes caused by each changing input.

This is called the **total differential** $\mathrm{d}f$:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z + \cdots$$

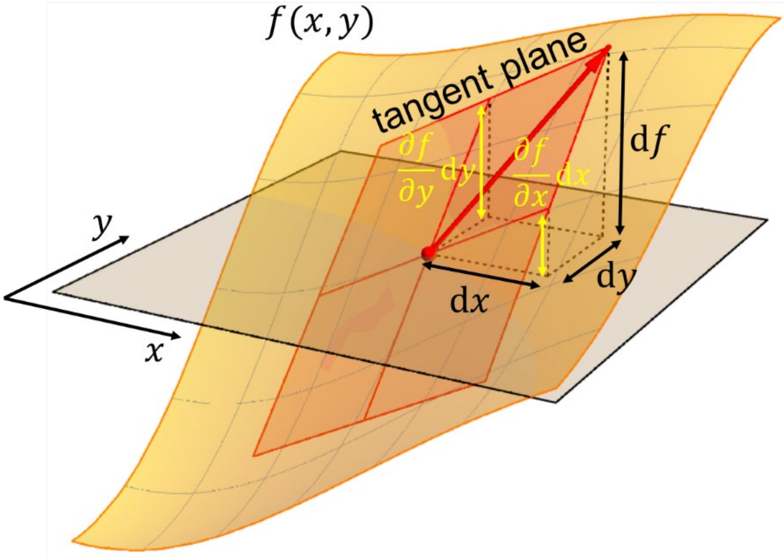
It tells us how much f changes as a function of how much we change the different inputs.

Let's consider the simple example $f(x, y)$ which you can imagine as the height of a mountainous terrain. Clearly at every point in this landscape, the "slope" (ratio of vertical to horizontal change) depends on which direction you are moving!



Since we are doing tiny steps, the mountainous terrain is locally considered as a flat plane (the tangent plane) so every possible direction's slope can be calculated by knowing the slope in just the two directions x- and y-, i.e. in terms of the two first order partial derivatives:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$



The total differential is, in fact, the z- component of the parametric equation of the tangent plane in 3D space, with $\mathrm{d}x$ and $\mathrm{d}y$ being the two free parameters, representing displacements in x and y:

$$\mathbf{r}_{\text{plane}}(\mathrm{d}x, \mathrm{d}y) = \underbrace{\begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}}_{\mathbf{r}_0} + \mathrm{d}x \underbrace{\begin{pmatrix} 1 \\ 0 \\ \partial f/\partial x \end{pmatrix}}_{\mathbf{v}_1} + \mathrm{d}y \underbrace{\begin{pmatrix} 0 \\ 1 \\ \partial f/\partial y \end{pmatrix}}_{\mathbf{v}_2} = \begin{pmatrix} x + \mathrm{d}x \\ y + \mathrm{d}y \\ f(x,y) + \mathrm{d}f(\mathrm{d}x, \mathrm{d}y) \end{pmatrix}$$

2) Find the total differential $\mathrm{d}f$ of $f(x,y) = x^2 + 3xy$.

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

The partial derivatives are: $\frac{\partial f}{\partial x} = 2x + 3y$ and $\frac{\partial f}{\partial y} = 3x$. Therefore:

$$\mathrm{d}f = (2x + 3y)\mathrm{d}x + (3x)\mathrm{d}y$$

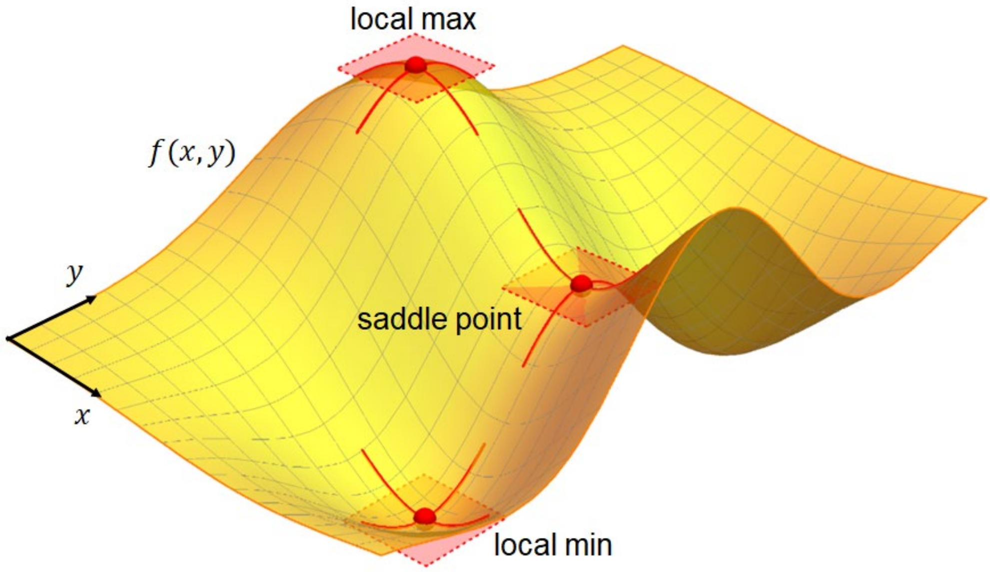
3) Find the total differential $\mathrm{d}f$ of $f(x,y) = ye^{x+y}$.

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

The partial derivatives are: $\frac{\partial f}{\partial x} = ye^{x+y}$ and $\frac{\partial f}{\partial y} = ye^{x+y} + e^{x+y}$. Therefore:

$$\mathrm{d}f = (ye^{x+y})\mathrm{d}x + (1 + y)e^{x+y}\mathrm{d}y$$

C. STATIONARY POINTS OF FUNCTIONS OF TWO VARIABLES



In single variable functions $f(x)$ the stationary points occur when $\mathrm{d}f = 0$ which happens when $\mathrm{d}f/\mathrm{d}x = 0$. For two-variable functions $f(x,y)$, the stationary points also occur when $\mathrm{d}f = 0$ in every direction, i.e. when the **tangent plane is horizontal** $\Leftrightarrow$ when **both first order partial derivatives are zero**:

$$\text{Stationary points} \quad \Leftrightarrow \quad \mathrm{d}f = 0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

The stationary points of $f(x,y)$ can be of three different types: **a local maximum, a local minimum, or a saddle point**. Saddle points are locally flat, but non-locally after some distance, f increases in some direction(s) but decreases in some other direction(s).

As with functions $f(x)$, this classification can be achieved by looking at second order derivatives. However, since now we have lots of different directions to move along, this all becomes a bit trickier. It is much easier to understand by considering the Taylor expansion of a 2D function $f(x,y)$.

TAYLOR EXPANSION OF A 2D FUNCTION

$$f(x,y) = f(x_0 + \Delta x, y_0 + \Delta y) =$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} \right)^n f(x,y) \right]_{x_0,y_0}$$

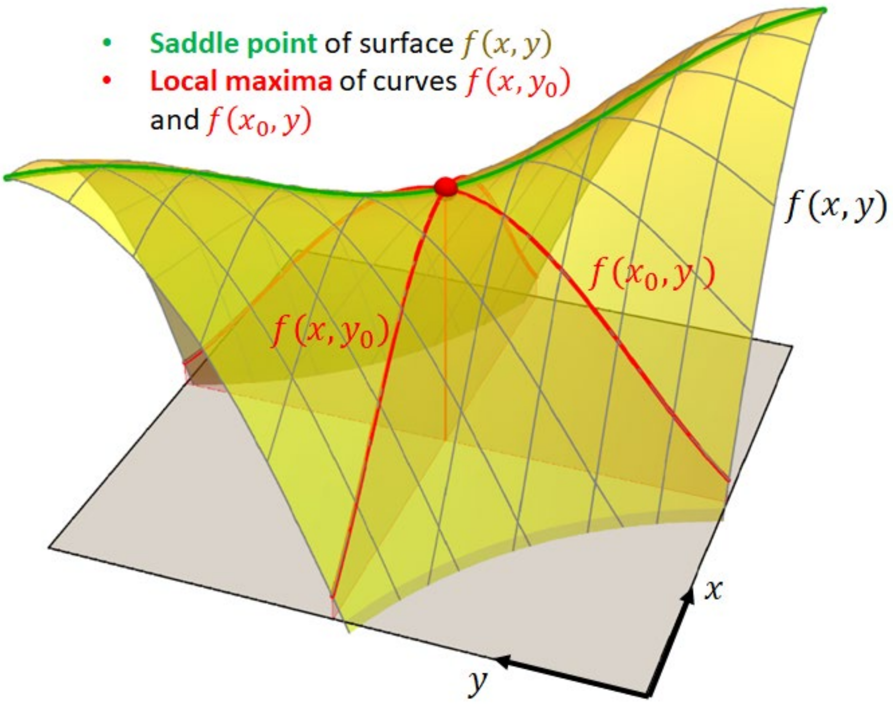
Expanding the bracket for the first three terms $n = 0,1,2$ explicitly, we have:

$$= \underbrace{f(x_0,y_0) + \underbrace{\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y}_{\text{first order } \Delta f(\Delta x, \Delta y)}}_{\text{equation of tangent plane}} + \underbrace{\frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\Delta x \Delta y) \right]}_{\substack{\text{second order correction } \Delta^{(2)} f(\Delta x, \Delta y) \\ \text{Note: } \Delta^{(2)} \text{ is my notation, not standard}}} + \cdots$$

Stationary values for function $f(x, y)$:

- Stationary point if $\Delta f = 0$ in **all directions** $\Leftrightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$
- Local maxima if $\Delta^{(2)} f < 0$ in **all directions**.
- Local minima if $\Delta^{(2)} f > 0$ in **all directions**.

Unfortunately, checking the sign of $\Delta^{(2)} f$ in all directions is not as simple as checking the sign of $\partial^2 f / \partial x^2$, $\partial^2 f / \partial y^2$ and $\partial^2 f / \partial x \partial y$ because the behaviour along the directions x and y might be different to the behaviour along some other direction!



Example: In the figure we show $f(x, y) = 1 - 7(x + y)^2 + (x - y)^2 = 1 - 3x^2 - 8xy - 3y^2$
The first partial derivatives are $\frac{\partial f}{\partial x} = -6x - 8y$ and $\frac{\partial f}{\partial y} = -8x - 6y$.
Both are zero at the origin $(x, y) = (0,0)$, which is therefore a stationary point. The second partial derivatives are:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = -6 \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = -6$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = -8 \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = -8$$

All the second order partial derivatives are negative… HOWEVER the point is not a local maximum. Consider the second order correction term in the Taylor expansion:

$$\Delta^{(2)} f(\Delta x, \Delta y) = \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\Delta x \Delta y)\right] = -3(\Delta x)^2 - 3(\Delta y)^2 - 8(\Delta x \Delta y)$$

It is not negative for every direction! For example, when we move along the diagonal direction $\Delta \mathbf{s} = (\Delta x, \Delta y) = (1, -1)$, then $\Delta^{(2)} f(\Delta x = -1, \Delta y = -1) = 2$ is positive, as clearly shown in the figure.

Therefore, to guarantee that we have a local minimum, we need to guarantee that: $\Delta^{(2)} f(\Delta x, \Delta y) = \frac{1}{2}\left(f_{xx}(\Delta x)^2 + f_{yy}(\Delta y)^2 + 2f_{xy}(\Delta x \Delta y)\right) > 0$ for **every possible combination of Δx and Δy** (i.e. for every direction). With some simple algebra, we can rearrange the second order correction to:

$$\Delta^{(2)} f(\Delta x, \Delta y) = \frac{1}{2}\left[f_{xx}\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right)(\Delta y)^2\right]$$

Which is easier to check for its sign as the squared terms are always positive. We only need to look at the non-squared terms. This gives us a recipe to classify stationary points:

Stationary points for function $f(x, y)$:

- Stationary point if $\Delta f = 0$ in **all directions** $\Leftrightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$
- $f_{xx}f_{yy} - f_{xy}^2 < 0 \Longrightarrow$ **Saddle point** (easy case: if f_{xx} and f_{yy} have opposite sign)
- $f_{xx}f_{yy} - f_{xy}^2 > 0 \Longrightarrow$ **Maximum** or **minimum**
  - Both f_{xx} and f_{yy} are positive $\Longrightarrow$ Local **minima**
  - Both f_{xx} and f_{yy} are negative $\Longrightarrow$ Local **maxima**
- $f_{xx}f_{yy} - f_{xy}^2 = 0 \Longrightarrow$ **Undetermined**. There is a direction where the function is flat to second order. Further investigation (higher order Taylor) is required.

This is all easy to derive from the rearranged second order correction, which I will give in the exam.

HIGHER NUMBER OF DIMENSIONS

(Not included in the exam, but I include it here for completeness)

The situation is more complex when the function has more than two input variables, but the essence is the same: we may consider the Taylor expansion $f = f_0 + \Delta f + \Delta^{(2)} f + \cdots$ with n-th order corrections and follow the same logic. Finding the sign of $\Delta^{(2)} f$ in all directions involves solving eigenvalues and eigenvectors!

For completeness, here is the **Taylor expansion in the general case** of M input variables, collected as an input vector $\mathbf{x}$:

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[(\Delta\mathbf{x} \cdot \nabla)^n f(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_0}$$

where ∇ is the **nabla operator**, defined as: $\nabla \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_M}\right)$, $\Delta\mathbf{x} \equiv (\Delta x_1, \Delta x_2, \dots, \Delta x_M)$ and $\Delta\mathbf{x} \cdot \nabla$ is a dot product between them.

4) Show that the function $f(x,y) = x^3 \exp(-x^2 - y^2)$ has a maximum at the point $(\sqrt{3/2}, 0)$, a minimum at $(-\sqrt{3/2}, 0)$, and a stationary point at the entire y-axis whose nature cannot be determined by the above procedures.

First, calculate the first partial derivatives (remember the product rule), and set them to zero to find the stationary points:

$$\frac{\partial f}{\partial x} = 3x^2 \exp(-x^2 - y^2) + x^3(-2x)\exp(-x^2 - y^2) = (3x^2 - 2x^4)\exp(-x^2 - y^2) = 0$$

$$\frac{\partial f}{\partial y} = -2yx^3 \exp(-x^2 - y^2) = 0$$

The second equation requires necessarily either $x = 0$ or $y = 0$. When $x = 0$, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ for all values of y, so the entire y axis is a stationary point. When $y = 0$, the first equation is zero at the locations $3x^2 - 2x^4 = 0 \rightarrow \quad x = \pm\sqrt{3/2}$, so the two points $(\pm\sqrt{3/2}, 0)$ are stationary.

Now we find the second partial derivatives to classify the stationary points (remember to use the product rule as needed):

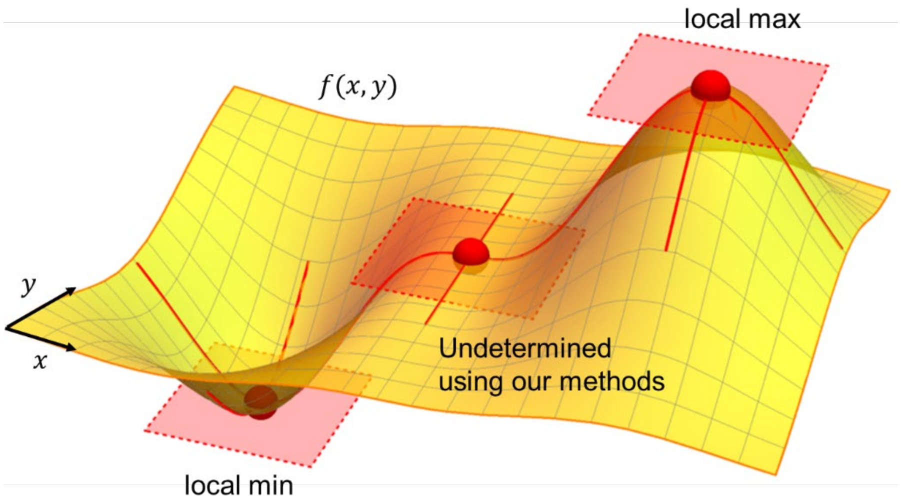
$$\begin{aligned} f_{xx} &= (4x^5 - 14x^3 + 6x)\exp(-x^2 - y^2), \\ f_{yy} &= x^3(4y^2 - 2)\exp(-x^2 - y^2) \\ f_{xy} &= 2x^2y(2x^2 - 3)\exp(-x^2 - y^2) \end{aligned}$$

If we substitute the pairs of values of x and y at $x = 0$ we get: $f_{xx} = f_{yy} = f_{xy} = 0$, so those points are undetermined! The function is flat to second order here, and further study would be required.

If we substitute the pairs of values of x and y at $(\pm\sqrt{3/2}, 0)$ we get:

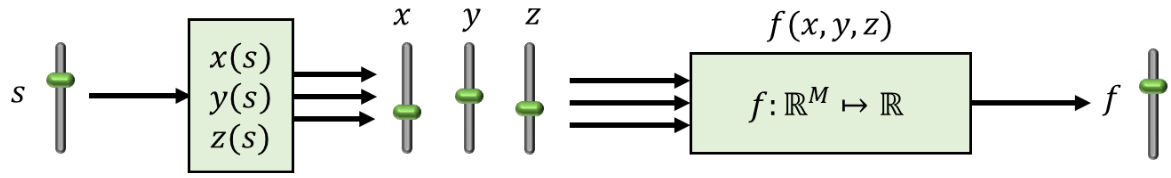
$$f_{xx} = \mp 6\sqrt{\frac{3}{2}}\exp\left(-\frac{3}{2}\right), \quad f_{yy} = \mp 3\sqrt{\frac{3}{2}}\exp\left(-\frac{3}{2}\right), \quad f_{xy} = 0$$

Applying the criteria, $f_{xx}f_{yy} - f_{xy}^2 > 0$, so that $(\sqrt{3/2}, 0)$ is a maximum (f_{xx} and $f_{yy} < 0$) and $(-\sqrt{3/2}, 0)$ is a minimum (f_{xx} and $f_{yy} > 0$). Here is an actual plot of the function.



D. CHAIN RULE

What if the variables $(x, y, z, \ldots)$ are themselves functions of a single variable s? This is represented in the following diagram:



We may want to obtain the **total derivative** of f with respect to s, i.e. how much the output f moves when we move the input knob s, this is the usual one-dimensional derivative of a single variable function $\mathrm{d}f/\mathrm{d}s$. One way to obtain it is to substitute $x(s)$, $y(s)$, etc. into the function $f(x, y, \ldots)$ and calculating the derivative in the traditional way. Another way is to use the chain rule, arrived at by simply dividing the total differential $(\mathrm{d}f)$ by the differential of the variable $(\mathrm{d}s)$:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z + \cdots$$

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s} + \cdots}$$

As a practical example, consider the energy of a mass m moving at speed v at a height h in a gravitational field g. The energy is the sum of the potential and kinetic energies: $E = mgh + \frac{1}{2}mv^2$. This expression has partial derivatives $\partial E/\partial h$ and $\partial E/\partial v$. However, the height h and the speed v might both depend on a single parameter, e.g. they are functions of time t. We might then want to calculate the rate of change of energy with respect to time $\mathrm{d}E/\mathrm{d}t$.

5) Given that $x(u) = 1 + au$ and $y(u) = bu^3$, find the rate of change of $f(x, y) = xe^{-y}$ with respect to u.

Solution: The partial derivatives of f are given by: $\frac{\partial f}{\partial x} = e^{-y}$ and $\frac{\partial f}{\partial y} = -xe^{-y}$. The chain rule therefore gives us the total derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}u} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}u} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}u} = (e^{-y})(a) + (-xe^{-y})(3bu^2)$$

Which after substituting $x = 1 + au$ and $y = bu^3$ gives:

$$\frac{\mathrm{d}f}{\mathrm{d}u} = ae^{-bu^3} - 3bu^2(1 + au)e^{-bu^3}$$

Note that we could have solved this exercise by brute force, directly obtaining $f(u) = f\big(x(u), y(u)\big) = (1 + au)e^{-bu^3}$ and finding the derivative $\mathrm{d}f/\mathrm{d}u$.

Notice how the famous "product rule" is simply a specific case of this general chain rule!

If $f(u, v) = uv$ with $u = u(x)$ and $v = v(x)$, then $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$

TOTAL VS PARTIAL DERIVATIVE

The exercise below introduces us to some interesting notation: In some contexts, it turns out that the variable s on which the inputs depend **may itself be one of the input variables** of the function. For example, consider a function $f(x,y,z)$ in which $y = y(x)$ and $z = z(x)$. Then, from the chain rule above:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}$$

Note that the left-hand side of this equation is the **total derivative** $\mathrm{d}f/\mathrm{d}x$, whilst the **partial derivative** $\partial f/\partial x$ appears as part of the right-hand side. In this case, the use of different symbols d and ∂ is helpful. **This is the reason that a different symbol is used for partial derivatives**. When evaluating this partial derivative, remember we must consider only the explicit appearances of x in the function f without using the knowledge that changing x necessarily changes y and z. The contribution from these latter changes is precisely accounted for by the other terms.

If you find this confusing, you can avoid all the confusion by using different variable names: you could use s for the global input variable on which x, y, z depend, and use the chain rule with $x(s) = s$.

6) Find the total derivative of $f(x,y) = x^2 + 3xy$ with respect to x given that $y = \sin^{-1} x$

Solution: The chain rule gives us:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \quad \left(\text{or, we could use } \frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} \text{ with } x = s\right)$$

On one hand, we need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$x = \sin y \rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

On the other hand, we need to find the partial derivatives of $f(x,y) = x^2 + 3xy$, calculated **always assuming that the other variable is constant**, without worrying for the fact that y changes with x.

$$\frac{\partial f}{\partial x} = 2x + 3y \qquad \frac{\partial f}{\partial y} = 3x$$

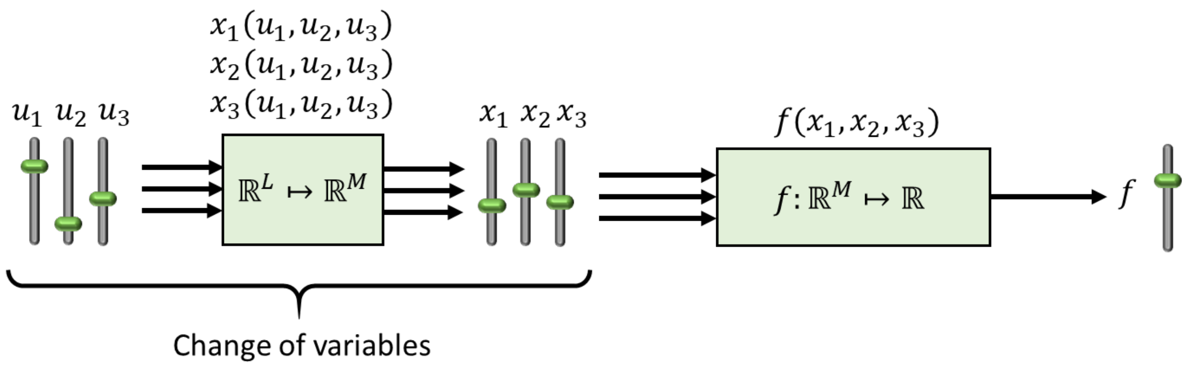
$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = (2x + 3y) + \frac{3x}{\sqrt{1-x^2}}$$

We can now substitute $y = \sin^{-1} x$ to get:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2x + 3\sin^{-1} x + \frac{3x}{\sqrt{1-x^2}}$$

E. CHANGE OF VARIABLES

What if the variables are themselves functions of **multiple** variables?



This is a **CHANGE OF VARIABLES**: for example – changing cartesian to cylindrical coordinates as input variables. The result is a new function of the new variables: $g(u_1,u_2,\dots,u_L) = f(x_1(u_1,u_2,\dots,u_L), x_2(u_1,u_2,\dots,u_L), \dots, x_M(u_1,u_2,\dots,u_L))$.

> In general, the number of variables need not be equal $M \neq L$.
>
> BUT when u's and x's are two sets of **independent variables**, then $M = L$.

Since this function represents the same transformation, we generally use the same symbol f (instead of a new symbol g) but we explicitly write the names of the input variables, e.g. $f(u_1,u_2,u_3)$ vs. $f(x_1,x_2,x_3)$. We may then want to know **how are the partial derivatives with respect to the old and new variables related**. The answer is simply to apply the chain rule for each variable. This time, everything is a partial derivative:

$$\begin{cases} \dfrac{\partial f}{\partial u_1} = \dfrac{\partial f}{\partial x_1}\dfrac{\partial x_1}{\partial u_1} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial x_2}{\partial u_1} + \cdots + \dfrac{\partial f}{\partial x_M}\dfrac{\partial x_M}{\partial u_1} \\ \dfrac{\partial f}{\partial u_2} = \dfrac{\partial f}{\partial x_1}\dfrac{\partial x_1}{\partial u_2} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial x_2}{\partial u_2} + \cdots + \dfrac{\partial f}{\partial x_M}\dfrac{\partial x_M}{\partial u_2} \\ \vdots \\ \dfrac{\partial f}{\partial u_L} = \dfrac{\partial f}{\partial x_1}\dfrac{\partial x_1}{\partial u_L} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial x_2}{\partial u_L} + \cdots + \dfrac{\partial f}{\partial x_M}\dfrac{\partial x_M}{\partial u_L} \end{cases}$$

In summation notation:

$$\frac{\partial f}{\partial u_j} = \sum_{i=1}^{M} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial u_j} \qquad \text{for} \quad j = 1,2,\dots,L$$

7) The 2D wave equation is $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ written for a function $f(x,y,t)$ in cartesian coordinates (x,y) and time t. Rewrite the wave equation using polar coordinates (ρ,ϕ). This equation is very common in physics and having it in polar form is even more common. The polar form is difficult to remember, and we usually look it up… but for once, let's derive it here using the chain rule. (This problem is too messy to ask for in an exam. But it is educational.)

Solution: The equations for the change of variables (in both directions) are given by:

$$\begin{cases} x=\rho\cos\phi \\ y=\rho\sin\phi \end{cases} \rightarrow \begin{cases} \rho=(x^2+y^2)^{1/2} \\ \phi=\tan^{-1}(y/x) \end{cases}$$

Notice that the time variable t is unchanged and will not be involved in the change of variables. Before we start. Some useful simplifications are (using trigonometry):

$$\rightarrow \begin{cases} \cos\phi=x/\rho \\ \sin\phi=y/\rho \\ x^2+y^2=\rho^2 \end{cases}$$

We need to find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of cylindrical coordinates. For that purpose, we use the change of variables chain rule:

$$\begin{cases} \dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial \rho}\dfrac{\partial \rho}{\partial x}+\dfrac{\partial f}{\partial \phi}\dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial f}{\partial y}=\dfrac{\partial f}{\partial \rho}\dfrac{\partial \rho}{\partial y}+\dfrac{\partial f}{\partial \phi}\dfrac{\partial \phi}{\partial y} \end{cases}$$

The four partial derivatives between coordinates (ρ,ϕ) and (x,y) are (remember derivative of $\tan^{-1}u(x)=u_x/(1+u^2)$ using the notation $u_x\equiv\partial u/\partial x$):

$$\frac{\partial\rho}{\partial x}=\frac{x}{(x^2+y^2)^{1/2}}=\cos\phi \qquad \frac{\partial\phi}{\partial x}=\frac{u_x}{1+u^2}\bigg|_{u=\frac{y}{x}}=\frac{-(y/x^2)}{1+(y/x)^2}=\frac{-y}{x^2+y^2}=-\frac{\sin\phi}{\rho}$$

$$\frac{\partial\rho}{\partial y}=\frac{y}{(x^2+y^2)^{1/2}}=\sin\phi \qquad \frac{\partial\phi}{\partial y}=\frac{u_y}{1+u^2}\bigg|_{u=\frac{y}{x}}=\frac{(1/x)}{1+(y/x)^2}=\frac{x}{x^2+y^2}=\frac{\cos\phi}{\rho}$$

So we substitute into the chain rule:

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}=\cos\phi\frac{\partial f}{\partial\rho}-\frac{\sin\phi}{\rho}\frac{\partial f}{\partial\phi}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y}+\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}=\sin\phi\frac{\partial f}{\partial\rho}+\frac{\cos\phi}{\rho}\frac{\partial f}{\partial\phi}$$

Since this is true for **any** function f, we can interpret the "partial derivative with respect to x and y" as operators:

$$\frac{\partial}{\partial x}=\cos\phi\frac{\partial}{\partial\rho}-\frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y}=\sin\phi\frac{\partial}{\partial\rho}+\frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}$$

Which can be applied twice to obtain the double derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)f$$

Start by applying the first (rightmost) operator to f. This step is trivial:

$$\frac{\partial^2 f}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial f}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial f}{\partial\phi}\right)$$

But be very careful in the next step: for instance, when we apply $\frac{\partial}{\partial\rho}$ to the term $-\frac{\sin\phi}{\rho}\frac{\partial f}{\partial\phi}$, **we need to use the product rule!** These are the four required terms fleshed out:

$$\left(\cos\phi\frac{\partial}{\partial\rho}\right)\cos\phi\frac{\partial f}{\partial\rho} = \cos^2\phi\frac{\partial^2 f}{\partial\rho^2}$$

$$\left(\cos\phi\frac{\partial}{\partial\rho}\right)\frac{-\sin\phi}{\rho}\frac{\partial f}{\partial\phi} = -\cos\phi\sin\phi\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right)\frac{\partial f}{\partial\phi} + \frac{1}{\rho}\frac{\partial^2 f}{\partial\rho\partial\phi}\right] = \cos\phi\sin\phi\left[\frac{1}{\rho^2}\frac{\partial f}{\partial\phi} - \frac{1}{\rho}\frac{\partial^2 f}{\partial\rho\partial\phi}\right]$$

$$\left(\frac{-\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)\cos\phi\frac{\partial f}{\partial\rho} = \frac{-\sin\phi}{\rho}\left[\frac{\partial}{\partial\phi}(\cos\phi)\frac{\partial f}{\partial\rho} + \cos\phi\frac{\partial^2 f}{\partial\rho\partial\phi}\right] = \frac{\sin^2\phi}{\rho}\frac{\partial f}{\partial\rho} - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2 f}{\partial\rho\partial\phi}$$

$$\left(\frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)\frac{\sin\phi}{\rho}\frac{\partial f}{\partial\phi} = \frac{\sin\phi}{\rho}\left[\frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{\rho}\right)\frac{\partial f}{\partial\phi} + \frac{\sin\phi}{\rho}\frac{\partial^2 f}{\partial\phi^2}\right] = \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial f}{\partial\phi} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}$$

Adding the four terms we get the final form for the double partial x derivative:

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\phi\frac{\partial^2 f}{\partial\rho^2} + \cos\phi\sin\phi\left[\frac{1}{\rho^2}\frac{\partial f}{\partial\phi} - \frac{1}{\rho}\frac{\partial^2 f}{\partial\rho\partial\phi}\right] + \frac{\sin^2\phi}{\rho}\frac{\partial f}{\partial\rho} - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2 f}{\partial\rho\partial\phi} + \frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial f}{\partial\phi} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \cos^2\phi\frac{\partial^2 f}{\partial\rho^2} + \frac{2\sin\phi\cos\phi}{\rho^2}\frac{\partial f}{\partial\phi} - \frac{2\sin\phi\cos\phi}{\rho}\frac{\partial^2 f}{\partial\rho\partial\phi} + \frac{\sin^2\phi}{\rho}\frac{\partial f}{\partial\rho} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}$$

And a similar long procedure for the y double partial derivative gives:

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\phi\frac{\partial^2 f}{\partial\rho^2} - \frac{2\sin\phi\cos\phi}{\rho^2}\frac{\partial f}{\partial\phi} + \frac{2\sin\phi\cos\phi}{\rho}\frac{\partial^2 f}{\partial\rho\partial\phi} + \frac{\cos^2\phi}{\rho}\frac{\partial f}{\partial\rho} + \frac{\cos^2\phi}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}$$

Adding both together, we can cancel some terms, and simplify others via $\cos^2\phi + \sin^2\phi = 1$, so:

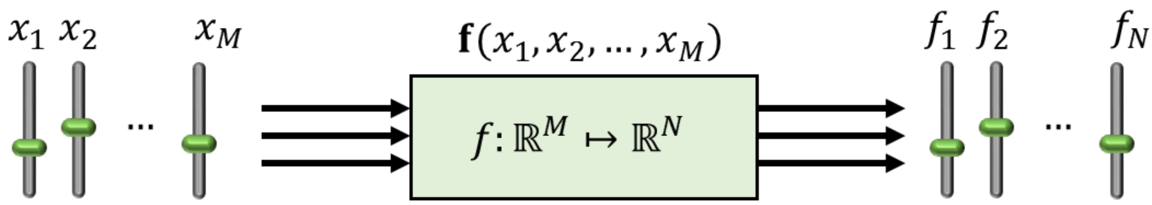
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}$$

It is interesting that **single derivatives appear in this expression** which originally involved only double derivatives. Also note that **the dimensions of the three terms are the same**, as they must be if we are adding them together (ρ and its differentials have dimensions of length, while ϕ and its differentials are dimensionless radians). So, finally, the 2D wave equation in polar coordinates may be written as:

$$\frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

F. DIFFERENTIATION WITH MULTIPLE INPUTS AND OUTPUTS: THE JACOBIAN MATRIX

Finally, we will generalize the concept of derivative to the most general case. A vector function with N input variables and M output components:



In this case, moving each input knob will move all output ones, so we can simply define a partial derivative for each input-to-output pair. All in all, there will be $N \times M$ first order partial derivatives, which can be arranged as a matrix, called the **Jacobian matrix**.

$$\mathbf{J} \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_M} \end{pmatrix}$$

We can write the total differential for each of the output components (considering each output as if it was a different function f_i) so we use the known equation for total differential:

$$\mathrm{d}f_i = \left(\frac{\partial f_i}{\partial x_1}\right)\mathrm{d}x_1 + \left(\frac{\partial f_i}{\partial x_2}\right)\mathrm{d}x_2 + \cdots + \left(\frac{\partial f_i}{\partial x_M}\right)\mathrm{d}x_M$$

telling us how much each output changes when we vary all the inputs. We can do this for all the output components of the function at the same time, by writing the vector total differential:

$$\mathbf{df} = \left(\frac{\partial \mathbf{f}}{\partial x_1}\right)\mathrm{d}x_1 + \left(\frac{\partial \mathbf{f}}{\partial x_2}\right)\mathrm{d}x_2 + \cdots + \left(\frac{\partial \mathbf{f}}{\partial x_M}\right)\mathrm{d}x_M$$

which can be conveniently written as a matrix-vector multiplication involving the Jacobian:

$$\mathbf{df} = \begin{pmatrix} \mathrm{d}f_1 \\ \mathrm{d}f_2 \\ \vdots \\ \mathrm{d}f_N \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)\mathrm{d}x_1 + \left(\frac{\partial f_1}{\partial x_2}\right)\mathrm{d}x_2 + \cdots + \left(\frac{\partial f_1}{\partial x_M}\right)\mathrm{d}x_M \\ \left(\frac{\partial f_2}{\partial x_1}\right)\mathrm{d}x_1 + \left(\frac{\partial f_2}{\partial x_2}\right)\mathrm{d}x_2 + \cdots + \left(\frac{\partial f_2}{\partial x_M}\right)\mathrm{d}x_M \\ \vdots \\ \left(\frac{\partial f_N}{\partial x_1}\right)\mathrm{d}x_1 + \left(\frac{\partial f_N}{\partial x_2}\right)\mathrm{d}x_2 + \cdots + \left(\frac{\partial f_N}{\partial x_M}\right)\mathrm{d}x_M \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_M} \end{pmatrix}}_{\text{Jacobian matrix } \mathbf{J}} \underbrace{\begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_M \end{pmatrix}}_{\mathbf{dx}} = \mathbf{J}\,\mathbf{dx}$$

This total differential tells us how much the function changes when we move each of the inputs by a small amount. It is the high dimensional analogy to the usual differentiation:

$f(x)(\mathbb{R} \to \mathbb{R})$	$\mathbf{f}(\mathbf{x})(\mathbb{R}^M \to \mathbb{R}^N)$
$x \to f$ $x + \mathrm{d}x \to f + \underbrace{\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x}_{\mathrm{d}f}$	$\mathbf{x} \to \mathbf{f}$ $\mathbf{x} + \mathbf{dx} \to \mathbf{f} + \underbrace{\mathbf{J}\,\mathbf{dx}}_{\mathbf{df}}$

LINEAR APPROXIMATIONS:

Differentiation always deals with infinitesimal increments around a point, such that $\mathrm{d}f = f'\,\mathrm{d}x$ is exact because, locally, the function is a straight line. However, the same equation becomes a good approximation (but not exact) when considering finite steps Δx if they are small.

Consider the linear approximation (first order Taylor expansion) of a one-dimensional function $f(x)$ around a point x_0:

$$f(x_0 + \Delta x) \approx \underbrace{f(x_0)}_{\text{constant}} + \underbrace{\frac{\mathrm{d}f}{\mathrm{d}x}\Delta x}_{\substack{\Delta f \\ \text{linear function} \\ \text{of } \Delta x}}$$

The Jacobian matrix generalizes the concept of the derivative, because it provides **the best linear approximation to a function at a point**. Any non-linear function $\mathbf{f}(\mathbf{x})$ with an arbitrary number of inputs and outputs can be **locally** approximated by a linear function:

$$\mathbf{f}(\mathbf{x}_0 + \Delta\mathbf{x}) \approx \underbrace{\mathbf{f}(\mathbf{x}_0)}_{\text{constant}} + \underbrace{\mathbf{J}\,\Delta\mathbf{x}}_{\substack{\Delta\mathbf{f} \\ \text{linear function} \\ \text{of } \Delta\mathbf{x}}}$$

8) Calculate the Jacobian matrix for the function $\mathbf{f}(x,y) = (x^2y, 5x + \sin y)$. Hence find the best linear approximation to the function at the point $(x,y) = (1,1)$.

Solution: We must calculate the four partial derivatives:

$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}(x^2y) = 2xy, \qquad \frac{\partial f_1}{\partial y} = \frac{\partial}{\partial y}(x^2y) = x^2$$

$$\frac{\partial f_2}{\partial x} = \frac{\partial}{\partial x}(5x + \sin y) = 5, \qquad \frac{\partial f_2}{\partial y} = \frac{\partial}{\partial y}(5x + \sin y) = \cos y$$

So the Jacobian matrix is: $\mathbf{J}(x,y) = \begin{pmatrix} 2xy & x^2 \\ 5 & \cos y \end{pmatrix}$.

The best linear approximation to the function is, at every point, given by:

$$\mathbf{f}(x_0 + \Delta x, y_0 + \Delta y) \approx \mathbf{f}(x_0, y_0) + \mathbf{J}(x_0, y_0)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \approx \begin{pmatrix} x_0^2 y_0 \\ 5x_0 + \sin y_0 \end{pmatrix} + \begin{pmatrix} 2x_0y_0 & x_0^2 \\ 5 & \cos y_0 \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

At the point $(x_0, y_0) = (1,1)$, the best linear approximation to the function is:

$$\mathbf{f}(1 + \Delta x, 1 + \Delta y) \approx \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 5 & \cos 1 \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 1 + 2\Delta x + \Delta y \\ 5 + 5\Delta x + \cos(1)\,\Delta y \end{pmatrix}$$

$$\mathbf{L}(x,y) = \begin{pmatrix} -2 + 2x + y \\ 5x + \cos(1)\,(y-1) \end{pmatrix}$$ after substituting $\Delta x = x - x_0$ and $\Delta y = y - y_0$

JACOBIAN DETERMINANT:

The determinant of the Jacobian matrix is often used (we will use it in next chapters) and is referred to as Jacobian determinant $\det(\mathbf{J})$, and most of the times (confusingly) as simply "the Jacobian". It is often written as $\frac{\partial(f_1, f_2, \ldots, f_N)}{\partial(x_1, x_2, \ldots, x_M)}$ to help us remember how to build the matrix.

G. FINDING THE BASIS VECTORS FOR ARBITRARY COORDINATE SYSTEMS

The conversion between coordinate systems is a typical example of a function with multiple inputs and multiple outputs.

$$\begin{cases} x_1 = x_1(u_1, u_2, u_3) \\ x_2 = x_2(u_1, u_2, u_3) \\ x_3 = x_3(u_1, u_2, u_3) \end{cases}$$

One common application of partial differentiation in this case, is to find a vector pointing in the direction in which each coordinate moves the position vector.

The basis vectors on arbitrary coordinates can be found by performing a differentiation of the vector with respect to the coordinate being changed:

$$\mathbf{e}_{u_i} = \frac{\partial \mathbf{r}}{\partial u_i}$$

The magnitude of these vectors is in general not equal to one.

We define the **scale factors** $h_i = \|\mathbf{e}_{u_i}\|$

Such that the unit vectors: $\hat{\mathbf{e}}_{u_i} = \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial u_i}$

9) Derive the unit-vector basis associated to cylindrical coordinates, and the scale factors:

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}$$

Solution: The basis vectors are defined in terms of the following vectors:

$$\mathbf{e}_\rho = \frac{\partial \mathbf{r}}{\partial \rho} = \left(\frac{\partial x}{\partial \rho}, \frac{\partial y}{\partial \rho}, \frac{\partial z}{\partial \rho}\right) = (\cos\phi, \sin\phi, 0)$$

$$\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi}\right) = (-\rho\sin\phi, \rho\cos\phi, 0)$$

$$\mathbf{e}_z = \frac{\partial \mathbf{r}}{\partial z} = \left(\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}\right) = (0,0,1)$$

The scale factors are the magnitude of these vectors: $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$.

Dividing the vectors $\mathbf{e}_i$ by their amplitude (the scale factors) we get the unit basis vectors:

$$\hat{\mathbf{e}}_\rho = \mathbf{e}_\rho = (\cos\phi, \sin\phi, 0)$$

$$\hat{\mathbf{e}}_\phi = \frac{\mathbf{e}_\phi}{\rho} = (-\sin\phi, \cos\phi, 0)$$

$$\hat{\mathbf{e}}_z = \mathbf{e}_z = (0,0,1)$$

Which are the well-known unit vectors in cylindrical coordinates.

10) Derive the unit-vector basis in spherical coordinates, and the scale factors:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

Solution: The basis vectors are defined in terms of the following vectors:

$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} = \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right) = (\sin\theta\cos\phi\,, \sin\theta\sin\phi\,, \cos\theta)$$

$$\mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) = (r\cos\phi\cos\theta\,, r\sin\phi\cos\theta\,, -r\sin\theta)$$

$$\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi}\right) = (-r\sin\theta\sin\phi\,, r\sin\theta\cos\phi\,, 0)$$

We can find the amplitude of these vectors:

$$h_r = |\mathbf{e}_r| = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2} = 1$$

$$h_\theta = |\mathbf{e}_\theta| = \sqrt{(r\cos\phi\cos\theta)^2 + (r\sin\phi\cos\theta)^2 + (r\sin\theta)^2} = r$$

$$h_\phi = |\mathbf{e}_\phi| = \sqrt{(r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2} = r\sin\theta$$

So, we can obtain the unit vectors by normalizing:

$$\hat{\mathbf{e}}_\rho = \mathbf{e}_\rho = (\cos\phi\,, \sin\phi\,, 0)$$

$$\hat{\mathbf{e}}_\theta = \frac{\mathbf{e}_\theta}{r} = (\cos\phi\cos\theta\,, \sin\phi\cos\theta\,, -\sin\theta)$$

$$\hat{\mathbf{e}}_\phi = \frac{\mathbf{e}_\phi}{r\sin\theta} = (\sin\phi\,, \cos\phi\,, 0)$$

Note that the scale factors are useful when defining an infinitesimal vector displacement in general curvilinear coordinates, in terms of their unit vectors (use the definition of total differential on $\mathbf{r}$):

$$\begin{aligned} \mathrm{d}\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u_1}\mathrm{d}u_1 + \frac{\partial \mathbf{r}}{\partial u_2}\mathrm{d}u_2 + \cdots + \frac{\partial \mathbf{r}}{\partial u_M}\mathrm{d}u_M \\ &= \mathbf{e}_1\mathrm{d}u_1 + \mathbf{e}_2\mathrm{d}u_2 + \cdots + \mathbf{e}_M\mathrm{d}u_M \\ &= h_1\mathrm{d}u_1\hat{\mathbf{e}}_1 + h_2\mathrm{d}u_2\hat{\mathbf{e}}_2 + \cdots + h_M\mathrm{d}u_M\hat{\mathbf{e}}_M \end{aligned}$$

This will be useful in the chapter on multiple integration:

Coordinate System	**Scale Factors**	**Infinitesimal vector displacement**
Cartesian (x, y, z)	$h_x = 1$; $h_y = 1$; $h_z = 1$	$\mathrm{d}\mathbf{r} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}} + \mathrm{d}z\,\hat{\mathbf{z}}$
Cylindrical (ρ, ϕ, z)	$h_\rho = 1$; $h_\phi = \rho$; $h_z = 1$	$\mathrm{d}\mathbf{r} = \mathrm{d}\rho\,\hat{\mathbf{e}}_\rho + \rho\,\mathrm{d}\phi\,\hat{\mathbf{e}}_\phi + \mathrm{d}z\,\hat{\mathbf{e}}_z$
Spherical (r, θ, ϕ)	$h_r = 1$; $h_\theta = r$; $h_\phi = r\sin\theta$	$\mathrm{d}\mathbf{r} = \mathrm{d}r\,\hat{\mathbf{e}}_r + r\,\mathrm{d}\theta\,\hat{\mathbf{e}}_\theta + r\sin\theta\,\mathrm{d}\phi\,\hat{\mathbf{e}}_\phi$

PROBLEMS:

11) Find all first and second partial derivatives of the function $f(x,y) = 2xe^y + yx$

First order:

$$\frac{\partial f}{\partial x} = 2e^y + y \qquad \frac{\partial f}{\partial y} = 2xe^y + x$$

Second order:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 0 \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2xe^y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2e^y + 1 \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2e^y + 1$$

12) Find the total differential $\mathrm{d}f$ of $f(x,y) = \sin(x^2 y)$.

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

The partial derivatives are: $\frac{\partial f}{\partial x} = 2xy\cos(x^2 y)$ and $\frac{\partial f}{\partial y} = x^2\cos(x^2 y)$. Therefore:

$$\mathrm{d}f = 2xy\cos(x^2 y)\,\mathrm{d}x + x^2\cos(x^2 y)\,\mathrm{d}y$$

13) Find and classify the stationary points of $f(x,y) = 1 - 2x + x^2 + y^2$

First order partial derivatives must be zero at stationary points:

$$f_x = \frac{\partial f}{\partial x} = -2 + 2x = 0$$

$$f_y = \frac{\partial f}{\partial y} = 2y = 0$$

From the first equation, the solution is $x = 1$. From the second equation, the solution is $y = 0$. Both are simultaneously zero at the point $(x,y) = (1,0)$.

Now, let's classify the stationary point. The second order partial derivatives are:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2 \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0$$

To classify the stationary points, we need to look at the sign of the second order correction.

$$\Delta^{(2)} f(\Delta x, \Delta y) = \tfrac{1}{2}\left[f_{xx}(\Delta x)^2 + f_{yy}(\Delta y)^2 + 2f_{xy}(\Delta x \Delta y)\right] \qquad \text{(Eq. 1)}$$

Which, with some algebra, can be rearranged into the following form (given in the exam):

$$\Delta^{(2)} f(\Delta x, \Delta y) = \frac{1}{2}\left[f_{xx}\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right)(\Delta y)^2\right]$$

Looking at the sign of the non-squared terms, we can arrive at the conditions used to classify the stationary point:

- $f_{xx}f_{yy} - f_{xy}^2 < 0 \Longrightarrow$ **Saddle point** (easy case: if f_{xx} and f_{yy} have opposite sign)
- $f_{xx}f_{yy} - f_{xy}^2 > 0 \Longrightarrow$ **Maximum** or **minimum**:
  Both f_{xx} and f_{yy} are positive $\Longrightarrow$ Local **minima**
  Both f_{xx} and f_{yy} are negative $\Longrightarrow$ Local **maxima**
- $f_{xx}f_{yy} - f_{xy}^2 = 0 \Longrightarrow$ **Undetermined**. There is a direction where the function is flat to second order. Further investigation (higher order Taylor) is required.

At $(x, y) = (1,0)$ we have $f_{xx} = 2$, $f_{yy} = 2$ and $f_{xy} = 0$. Therefore, $f_{xx}f_{yy} - f_{xy}^2 = 4 - 0 > 0$, so it is a maximum or minimum. Since f_{xx} and $f_{yy} > 0$, it is a local minimum.

14) Find and classify the stationary points of $f(x, y) = xy^2 + y - x$

First order partial derivatives must be zero at stationary points:

$$f_x = \frac{\partial f}{\partial x} = y^2 - 1 = 0 \qquad\qquad f_y = \frac{\partial f}{\partial y} = 2xy + 1 = 0$$

From the first equation we get $y = \pm 1$, and substituting it into the second, we get:

For $y = 1 \;\rightarrow\; 2x + 1 = 0 \;\rightarrow\; x = -\frac{1}{2}$

For $y = -1 \;\rightarrow\; -2x + 1 = 0 \;\rightarrow\; x = \frac{1}{2}$

So, the two stationary points are: $(x, y) = (-\frac{1}{2}, 1)$ and $(\frac{1}{2}, -1)$.

Now, let's classify them:

The second order partial derivatives are:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 0 \qquad\qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2xe^y$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2e^y + 1$$

To classify the stationary points, we need to look at the sign of the second order correction.

$$\Delta^{(2)} f(\Delta x, \Delta y) = \tfrac{1}{2}\left[f_{xx}(\Delta x)^2 + f_{yy}(\Delta y)^2 + 2f_{xy}(\Delta x \Delta y)\right] \qquad \text{(Eq. 1)}$$

Which, with some algebra, can be rearranged into the following form (given in the exam):

$$\Delta^{(2)} f(\Delta x, \Delta y) = \frac{1}{2}\left[f_{xx}\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right)(\Delta y)^2\right]$$

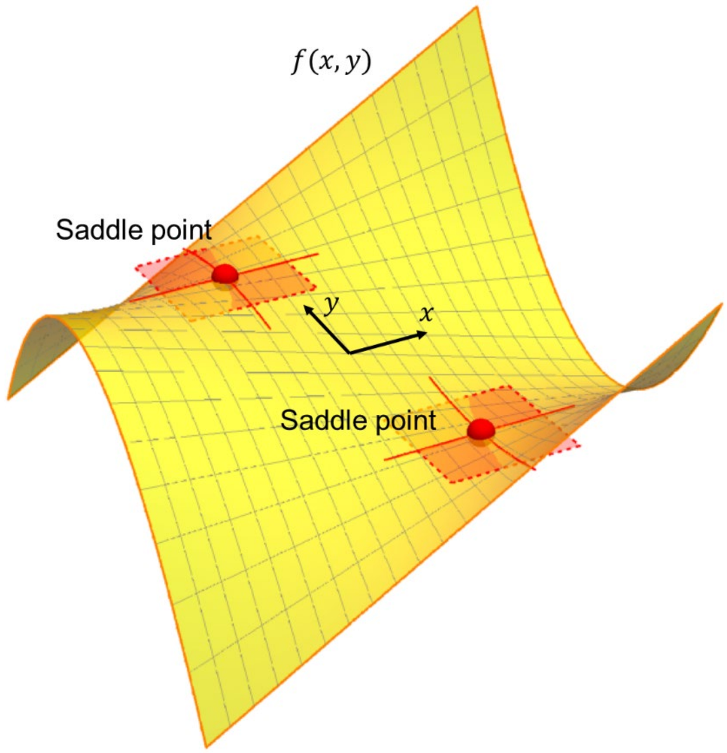
This problem introduces a minor setback, because here we have $f_{xx} = 0$ which means that this rearranged form has undetermined terms. We can however reason as follows: By looking at the original form of $\Delta^{(2)} f$ (Eq. 1), we see that it is completely symmetric in f_{xx} and in f_{yy}, therefore, we can deduce that an equivalent rearranged form can be arrived at by switching the roles of x and y, as follows:

$$\Delta^{(2)} f(\Delta x, \Delta y) = \frac{1}{2}\left[f_{yy}\left(\Delta y + \frac{f_{xy}}{f_{yy}}\Delta x\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{yy}}\right)(\Delta x)^2\right]$$

Which is now well defined. Looking at the sign of the non-squared terms, we can arrive exactly at the **same** conditions that we have previously used. In fact, **the conditions are valid in general**.

- $f_{xx}f_{yy} - f_{xy}^2 < 0 \Longrightarrow$ **Saddle point** (easy case: if f_{xx} and f_{yy} have opposite sign)
- $f_{xx}f_{yy} - f_{xy}^2 > 0 \Longrightarrow$ **Maximum** or **minimum**:
  Both f_{xx} and f_{yy} are positive $\Longrightarrow$ Local **minima**
  Both f_{xx} and f_{yy} are negative $\Longrightarrow$ Local **maxima**
- $f_{xx}f_{yy} - f_{xy}^2 = 0 \Longrightarrow$ **Undetermined**. There is a direction where the function is flat to second order. Further investigation (higher order Taylor) is required.

At $(x, y) = (-\frac{1}{2}, 1)$ we have $f_{xx} = 0$, $f_{yy} = -e$ and $f_{xy} = 2e + 1$.
$f_{xx}f_{yy} - f_{xy}^2 = 0 - (2e + 1) < 0$, therefore, it is a saddle point.
At $(x, y) = (\frac{1}{2}, -1)$ we have $f_{xx} = 0$, $f_{yy} = e^{-1}$ and $f_{xy} = 2e^{-1} + 1$.
$f_{xx}f_{yy} - f_{xy}^2 = 0 - (2e^{-1} + 1) < 0$, therefore, it is also a saddle point.



15) Find and classify the stationary points of $f(x, y) = xy + e^{xy}$

First order partial derivatives must be zero at stationary points:

$$f_x = \frac{\partial f}{\partial x} = y(1 + e^{xy}) = 0 \qquad\qquad f_y = \frac{\partial f}{\partial y} = x(1 + e^{xy}) = 0$$

From the first equation we get $y = 0$, and substituting it into the second, we get $x = 0$:

So the only stationary points is: $(x, y) = (0,0)$

Now, let's classify it:

The second order partial derivatives are:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = e^{xy}y^2 \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = e^{xy}x^2$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 1 + e^{xy}(1 + xy)$$

At the stationary point $(x, y) = (0,0)$ these take the values:

$$f_{xx} = 0, \quad f_{yy} = 0, \quad f_{xy} = 2$$

To classify the stationary points, we need to look at the sign of the second order correction in all directions.

$$\Delta^{(2)} f(\Delta x, \Delta y) = \frac{1}{2}\left[f_{xx}(\Delta x)^2 + f_{yy}(\Delta y)^2 + 2f_{xy}(\Delta x \Delta y)\right] \qquad \text{(Eq. 1)}$$

In this case we don't need to do work with any rearranged term, because we have $f_{xx} = f_{yy} = 0$ which leaves:

$$\Delta^{(2)} f(\Delta x, \Delta y) = 2\Delta x \Delta y$$

Clearly, the sign of the second order correction term depends on the signs of $\Delta x \Delta y$, and so it is positive or negative along different directions, so this is a saddle point.

We could also have checked the usual condition: $f_{xx}f_{yy} - f_{xy}^2 < 0 \Longrightarrow$ **Saddle point**

16) Given that $x(u) = au^2$ and $y(u) = bu^{-2}$, find the rate of change of $f(x, y) = x^2 e^{-y}$ with respect to u.

Solution: The partial derivatives of f are given by: $\frac{\partial f}{\partial x} = 2xe^{-y}$ and $\frac{\partial f}{\partial y} = -x^2 e^{-y}$. The derivatives of x and y are: $\frac{\mathrm{d}x}{\mathrm{d}u} = 2au$ and $\frac{\mathrm{d}y}{\mathrm{d}u} = -2bu^{-3}$ The chain rule therefore gives us the total derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}u} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}u} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}u} = (2xe^{-y})(2au) + (-x^2 e^{-y})(-2bu^{-3})$$

Which after substituting $x = au^2$ and $y = bu^{-2}$ gives:

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}u} &= \left(2au^2 e^{-bu^{-2}}\right)(2au) + \left(-a^2u^4 e^{-bu^{-2}}\right)(-2bu^{-3}) \\ &= 4a^2u^3 e^{-bu^{-2}} + 2a^2bue^{-bu^{-2}} \\ &= 2a^2ue^{-bu^{-2}}(2u^2 + b)\end{aligned}$$

Note that we could have solved this exercise by brute force, directly obtaining $f(u) = f\big(x(u), y(u)\big) = (au^2)^2 e^{-(bu^{-2})}$ and finding the derivative $\mathrm{d}f/\mathrm{d}u$.

17) Find the total derivative of $f(x,y) = x^2 + \ln(xy)$ with respect to x given that $y = e^x + x^2$

Solution: The chain rule gives us:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \quad \left(\text{or, we could use } \frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} \text{ with } x = s\right)$$

On one hand, we need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x + 2x$$

On the other hand, we need to find the partial derivatives of $f(x,y) = x^2 + \ln(xy)$, calculated **always assuming that the other variable is constant**, without worrying for the fact that y changes with x.

Remember the derivative of $\ln(u(x))$ is as follows: $\frac{\mathrm{d}}{\mathrm{d}x}(\ln u(x)) = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u'(x)}{u(x)}$

$$\frac{\partial f}{\partial x} = 2x + \frac{1}{x} \qquad \frac{\partial f}{\partial y} = \frac{1}{y}$$

Therefore, the chain rule tells us:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = \left(2x + \frac{1}{x}\right) + \left(\frac{1}{y}\right)(e^x + 2x)$$

We can now substitute $y = e^x + x^2$ to get:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2x + \frac{1}{x} + \frac{e^x + 2x}{e^x + x^2}$$

18) Consider the function $f(x,y) = x^2 + e^{xy}$ and the change of variables:

$$\begin{cases} u = x + y \\ v = x - y \end{cases}$$

Find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

Solution: We apply the chain rule (generalised version) to find $\frac{\partial f}{\partial u}$:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Before we start finding the partial derivatives, we need to know the “inverted” change of variables (i.e. x and y in terms of u and v). We can figure it out by solving the simultaneous equations for x and y as follows:

$$\begin{cases} x = (u + v)/2 \\ y = (u - v)/2 \end{cases}$$

Therefore, we can now calculate:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = (2x + ye^{xy})\left(\frac{1}{2}\right) + (xe^{xy})\left(\frac{1}{2}\right) = x + \frac{(x+y)}{2}e^{xy}$$

And substituting $x = (u + v)/2$ and $y = (u - v)/2$ we get:

$$\frac{\partial f}{\partial u} = \frac{u+v}{2} + \frac{u}{2} e^{\frac{u^2-v^2}{4}}$$

Next, we do the same for $\frac{\partial f}{\partial v}$:

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = (2x + ye^{xy})\left(\frac{1}{2}\right) + (xe^{xy})\left(-\frac{1}{2}\right) = x + \frac{(y-x)}{2}e^{xy}$$

And substituting $x = (u+v)/2$ and $y = (u-v)/2$ we get:

$$\frac{\partial f}{\partial u} = \frac{u+v}{2} - \frac{v}{2} e^{\frac{u^2-v^2}{4}}$$

As usual with these problems, this could have also been solved by brute force, substituting x and y as functions of u and v in the full expression of $f(x,y)$ and then finding the partial derivatives directly.

19) Consider the following change of coordinates (x,y) to (u,v):

$$\begin{cases} u = x + y \\ v = x - y \end{cases}$$

Find the unit vectors associated to the new coordinates, $\hat{\mathbf{e}}_u$ and $\hat{\mathbf{e}}_v$.

We need the conversion from (u,v) to (x,y), which can be obtained by solving the simultaneous equations for x and y:

$$\begin{cases} x = (u+v)/2 \\ y = (u-v)/2 \end{cases}$$

The unit vectors are given by how the position vector $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ changes when we change the coordinates u and v:

$$\mathbf{e}_u = \frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\mathbf{e}_v = \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right)$$

Dividing by their norm we can obtain the corresponding unit vectors:

$$\hat{\mathbf{e}}_u = \frac{\mathbf{e}_u}{\|\mathbf{e}_u\|} = \frac{1}{\sqrt{2}}(1,1)$$

$$\hat{\mathbf{e}}_v = \frac{\mathbf{e}_u}{\|\mathbf{e}_v\|} = \frac{1}{\sqrt{2}}(1,-1)$$

Note that this change of coordinates corresponds to a rotated (and scaled) rectangular grid.

20) Consider the vector function of two variables:

$$\mathbf{F}(x,y) = (xy^2 + y - x)\hat{\mathbf{x}} + (xy + e^{xy})\hat{\mathbf{y}}$$

Find the linear function $\mathbf{L}(x,y)$ which best approximates $\mathbf{F}(x,y)$ at the point $(x_0, y_0) = (0,1)$.

Solution: $\mathbf{F}(x,y)$ is a function with two inputs and two outputs. To find the best linear approximation, we can use the Jacobian matrix. The best linear approximation is given by:

$$\mathbf{L}(x_0 + \Delta x, y_0 + \Delta y) = \mathbf{F}(x_0, y_0) + \mathbf{J}(x_0, y_0)\, \Delta\mathbf{r}$$

The Jacobian matrix is:

$$\mathbf{J}(x,y) = \begin{pmatrix} \dfrac{\partial F_x}{\partial x} & \dfrac{\partial F_x}{\partial y} \\ \dfrac{\partial F_y}{\partial x} & \dfrac{\partial F_y}{\partial y} \end{pmatrix} = \begin{pmatrix} y^2 - 1 & 2xy + 1 \\ y + ye^{xy} & x + xe^{xy} \end{pmatrix}$$

Therefore, we know that the best linear approximation at every point is given by:

$$\mathbf{L}(x_0 + \Delta x, y_0 + \Delta y) = \begin{pmatrix} x_0 y_0^2 + y_0 - x_0 \\ x_0 y_0 + e^{x_0 y_0} \end{pmatrix} + \begin{pmatrix} y_0^2 - 1 & 2x_0 y_0 + 1 \\ y_0 + y_0 e^{x_0 y_0} & x_0 + x_0 e^{x_0 y_0} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

If we compute this linear function at the point $(x_0, y_0) = (0,1)$ we get:

$$\mathbf{L}(\Delta x, 1 + \Delta y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 1 + \Delta y \\ 1 + 2\Delta x \end{pmatrix}$$

We are asked for $\mathbf{L}(x,y)$ written as a function of x and y instead of Δx and Δy, so we can simply substitute $\Delta x = x - x_0 = x$ and $\Delta y = y - y_0 = y - 1$, and finally arrive at the linear function:

$$\mathbf{L}(x,y) = \begin{pmatrix} y \\ 1 + 2x \end{pmatrix} = y\hat{\mathbf{x}} + (1 + 2x)\hat{\mathbf{y}}$$

Which is, indeed, the best linear approximation to $\mathbf{F}(x,y)$ near the point $(0,1)$.

21) Prove that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial z}$ for any function $f(r,\theta,\phi)$ can be written using only spherical coordinates and partial derivatives with respect to spherical coordinates as:

$$\frac{\partial f}{\partial x} = \sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial f}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\frac{\partial f}{\partial z} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

Hint: spherical coordinate transformation:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

Hint: remember the derivative $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}u(x) = \frac{u'(x)}{u^2(x)+1}$

Solution:

For $\frac{\partial f}{\partial x}$, we apply the generalized chain rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

The required partial derivatives are:

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{r\sin\theta\cos\phi}{r} = \sin\theta\cos\phi$$

$$\frac{\partial \theta}{\partial x} = \frac{\frac{\partial}{\partial x}\left(\frac{\sqrt{x^2+y^2}}{z}\right)}{\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2+1} = \frac{\frac{1}{z}\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}}{\frac{x^2+y^2+z^2}{z^2}} = \frac{x\,z}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}$$
$$= \frac{r^2\sin\theta\cos\phi\cos\theta}{\sqrt{r^2(\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi)}}\frac{1}{r^2} = \frac{\sin\theta\cos\phi\ \cos\theta}{r\sin\theta} = \frac{\cos\phi\cos\theta}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{\frac{\partial}{\partial x}\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2+1} = \frac{-y/x^2}{\frac{x^2+y^2}{x^2}} = \frac{-y}{x^2+y^2}$$
$$= \frac{-r\sin\theta\sin\phi}{r^2(\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi)} = \frac{-r\sin\theta\sin\phi}{r^2\sin^2\theta} = -\frac{\sin\phi}{r\sin\theta}$$

Therefore,

$$\frac{\partial f}{\partial x} = (\sin\theta\cos\phi)\frac{\partial f}{\partial r} + \left(\frac{\cos\phi\cos\theta}{r}\right)\frac{\partial f}{\partial \theta} - \left(\frac{\sin\phi}{r\sin\theta}\right)\frac{\partial f}{\partial \phi}$$

For $\frac{\partial f}{\partial z}$, we apply the generalized chain rule:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}$$

The required partial derivatives are:

$$\frac{\partial r}{\partial z} = \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial \theta}{\partial z} = \frac{\frac{\partial}{\partial z}\left(\frac{\sqrt{x^2+y^2}}{z}\right)}{\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2+1} = \frac{-\sqrt{x^2+y^2}\,\frac{1}{z^2}}{\frac{x^2+y^2+z^2}{z^2}} = \frac{-\sqrt{x^2+y^2}}{(x^2+y^2+z^2)}$$
$$= -\frac{\sqrt{r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi)}}{r^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \phi}{\partial z} = \frac{\frac{\partial}{\partial z}\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2+1} = 0$$

Therefore:

$$\frac{\partial f}{\partial z} = (\cos\theta)\frac{\partial f}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial f}{\partial \theta}$$

This seemingly ugly expression makes intuitive sense if you think about it. It tells you how much the function changes along z as a function of how much it changes along r and θ.

For example, if $\theta = \frac{\pi}{2}$, we are in the XY plane, and then $\frac{\partial f}{\partial z} = 0\frac{\partial f}{\partial r} - \left(\frac{1}{r}\right)\frac{\partial f}{\partial \theta}$. Indeed, $\frac{\partial f}{\partial r}$ is irrelevant to $\frac{\partial f}{\partial z}$ when we are in the XY plane, while $\frac{\partial f}{\partial \theta}$ will contribute less to $\frac{\partial f}{\partial z}$ the bigger the radius is, and with a negative sign because an increase in θ will be a decrease in z.

On the contrary, if we are in the z-axis, with $\theta = 0$, then we have $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r} - 0\frac{\partial f}{\partial \theta}$. Indeed, when we are in the z-axis, $\frac{\partial f}{\partial z}$ must be the same thing as $\frac{\partial f}{\partial r}$, because in the z-axis the distance to the origin is exactly the z-coordinate directly.

4. MULTIPLE INTEGRATION

4.1 INTRODUCTION TO MULTIPLE INTEGRALS

We now focus on integration when a function has N variables as the input: interesting!

A. REVISION OF DEFINITE INTEGRALS

A definite integral gives us the area of the graph of a function $f(x)$ between $x = a$ and $x = b$:

$$I = \int_a^b f(x)\mathrm{d}x = \lim_{\Delta x \to 0} \left(\sum_{i=1}^{N} \Delta x \cdot f(x_i) \right)$$

It is the limit of a sum when the interval $[a, b]$ is divided by vertical lines into N stripes of the same width and of a height given by the value of the function inside the integral (called the **integrand**) at any point x_i within the stripe.

Important: after doing the integral, the variable being integrated disappears from the result:

$$I(y, z) = \int_a^b f(x, y, z)\mathrm{d}x$$

However, if the upper limit of the integral is considered as a variable x, then the result of the integral depends on x (after all, it is describing the area of $f(x')$ as a function of the upper limit) and therefore results in a function of x:

$$F(x) = \int_a^x f(x')\mathrm{d}x'$$

It is always a good idea to use a different variable name inside the integral (x'), to formally distinguish it from the variable in the limit (x), on which the result depends. If we differentiate this function $F(x)$, we recover the function inside the integral again. This result is called **The Fundamental Theorem of Calculus**:

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(x')\mathrm{d}x' \right) = f(x)$$

It tells us that **integration is inverse to differentiation**. This allows us to calculate integrals by finding an antiderivative (i.e. a function whose derivative is the integrand).

B. INTRODUCTION TO DOUBLE INTEGRALS

Double integrals can be done to functions which depend on at least two variables $f(x,y)$. Double integrals represent the volume under a graph of the function, limited to a certain area A in the two-dimensional space of the variables (x,y).

$$V = \iint_A f(x,y)\ \mathrm{d}A$$

INTERPRETATION AS THE LIMIT OF A SUM

Similarly to the case of the single integral, the double integral is defined as the limit of the sum of N small segments of volume whose bottom area is ΔA_i and whose height is given by the function $f(x_i,y_i)$ evaluated at any point (x_i,y_i) inside the area ΔA_i. The collection of all segment areas ΔA_i for $i=1$ to N covers the entire area A in the (x,y) plane.

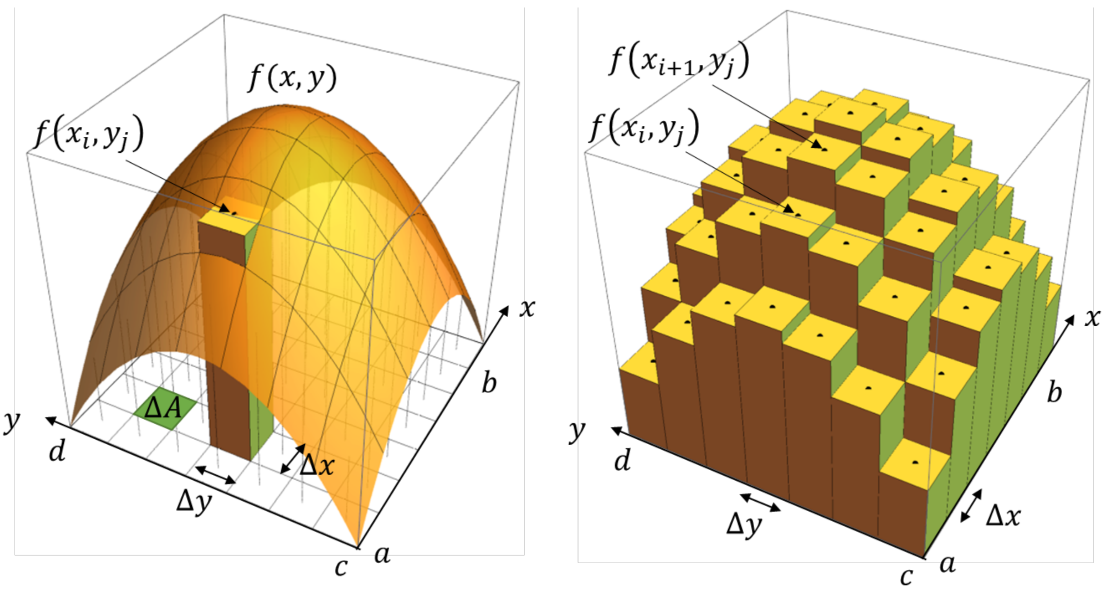
$$\iint_A f(x,y)\ \mathrm{d}A = \lim_{\Delta A\to 0}\left(\sum_{i=1}^{N} \Delta A_i \cdot f(x_i,y_i)\right)$$

For single integrals, the division of the interval $[a,b]$ into differential segments $\mathrm{d}x$ can only be done in one way. For double integrals, the division of the area A into differential segments $\mathrm{d}A$ can be done in many ways.

As a start, we will study **integration in rectangular coordinates**: using a rectangular grid of infinitesimal squares of sides $\mathrm{d}x$ and $\mathrm{d}y$, such that $\mathrm{d}A = \mathrm{d}x\ \mathrm{d}y$.

The integral is a limit of a double sum of cuboids with base being a tiny square of area $\Delta A = \Delta x \Delta y$ and height $f(x_i,y_j)$, where the double sum uses the indices i and j to label the rectangles across the x and y directions:

$$\iint_A f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \lim_{\Delta A\to 0}\left(\sum_{j=1}^{N_y}\sum_{i=1}^{N_x} \Delta x \cdot \Delta y \cdot f(x_i,y_i)\right)$$



DOUBLE INTEGRALS IN RECTANGULAR REGION: ITERATED INTEGRALS

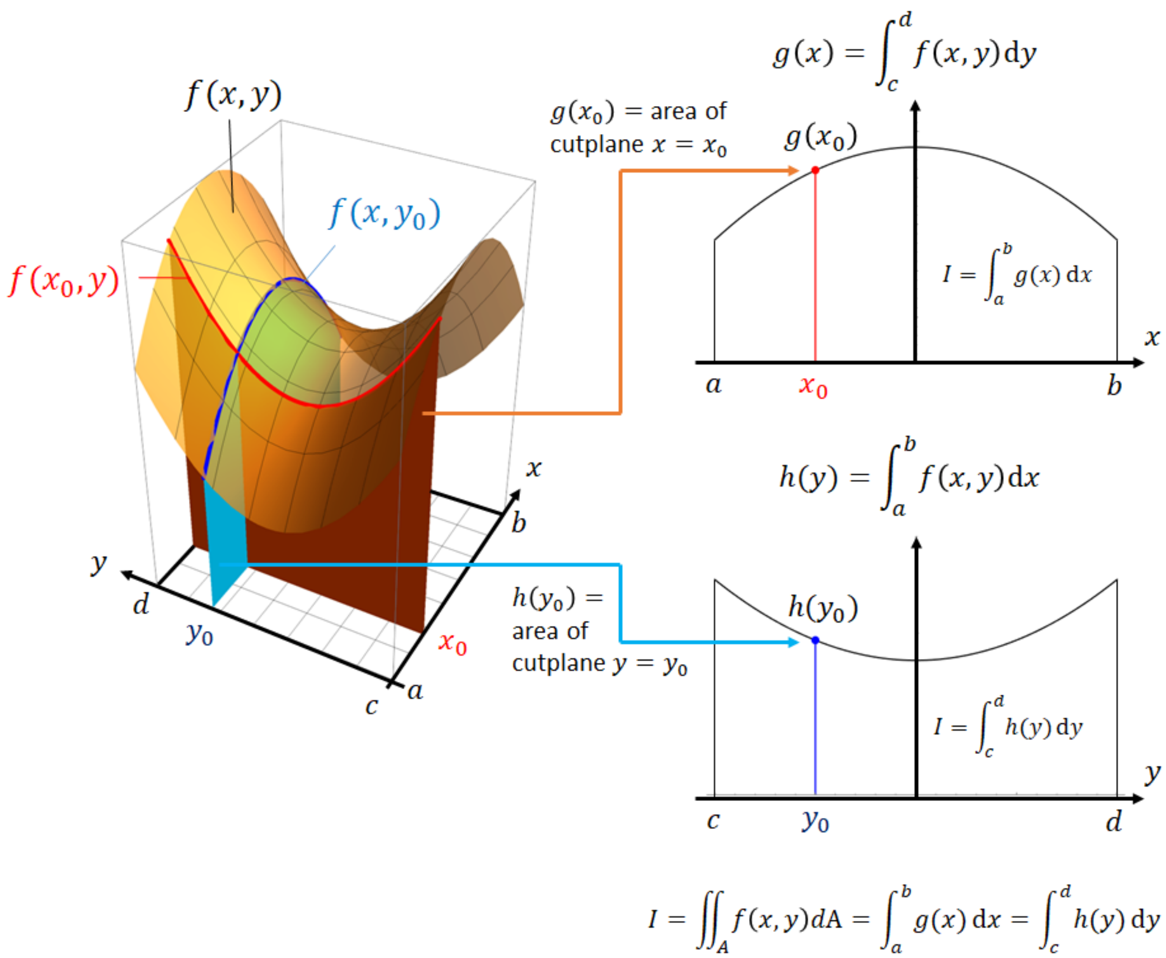
Double integrals can be calculated by simply considering one definite integral after another, and this may be done in any order. As a start, let's consider the simplest case in which region A is a rectangular region bounded by $x \in [a, b]$ and $y \in [c, d]$ so we simply perform the two integrals, in any order. This is called **Fubini's Theorem**:

$$I = \iint_A f(x, y)\, \mathrm{d}A = \int_c^d \left(\int_a^b f(x, y)\, \mathrm{d}x \right) \mathrm{d}y = \int_a^b \left(\int_c^d f(x, y)\, \mathrm{d}y \right) \mathrm{d}x$$

Think about the addition of cuboids before taking the limit: the order in which we sum the rectangular cuboids (whether we first add them along x, and then along y, or the other way around) is irrelevant to the final answer of the total volume.

Consider the first case above, where we first integrate over x and then over y. The inner integral $h(y) = \int_a^b f(x, y)\, \mathrm{d}x$ results in a function of y only. **Think about what this function represents: it is the area of a cut-plane of the original function.** Then, when we integrate $g(y)\, \mathrm{d}y$, the volume of a differential slice, in the interval $y \in [c, d]$, it results in the desired volume.

Visual outline of Fubini's theorem:



1) Calculate the following double integral, where the rectangular region A is bounded by $x \in [0,1]$ and $y \in [0,1]$

$$\iint_A x(y^2 + x)\,\mathrm{d}A$$

Solution: We do the division of the area into differentials of area using rectangular coordinates, so that $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, and perform the iterated integration in any order:

$$I = \iint_A x(y^2 + x)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\int_0^1 (xy^2 + x^2)\,\mathrm{d}x \right) \mathrm{d}y$$

The inner integral is:

$$h(y) = \int_0^1 (xy^2 + x^2)\,\mathrm{d}x = \frac{1}{2}y^2 + \frac{1}{3}$$

And it represents the area under $f(x, y)$ when y is fixed (that is why it is a function of y) and bounded between the two left and right limits $x \in [a, b]$.

The outer integral is then the final answer:

$$I = \int_0^1 h(y)\,\mathrm{d}y = \int_0^1 \left(\frac{1}{2}y^2 + \frac{1}{3} \right) \mathrm{d}y = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

Note that we obtain the same result if we had chosen the reverse order in the integration:

$$I = \iint_A x(y^2 + x)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \underbrace{\left(\int_0^1 (xy^2 + x^2)\,\mathrm{d}y \right)}_{h(x) = \frac{1}{3}x + x^2} \mathrm{d}x = \int_0^1 \left(\frac{1}{3}x + x^2 \right) \mathrm{d}x = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

SEPARATION OF ITERATED INTEGRALS

Often, the integrand can be written as a **product of two functions of a single variable each**. Then (as we know from standard integration) we can take out of the integral everything which does not depend on the integrated variable, as if it was a constant. Doing this, we can "separate" the double integral into the product of two integrals.

$$\iint_A f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_a^b f(x)g(y)\,\mathrm{d}x \right) \mathrm{d}y = \int_c^d g(y) \left(\int_a^b f(x)\,\mathrm{d}x \right) \mathrm{d}y$$
$$= \left(\int_a^b f(x)\,\mathrm{d}x \right) \left(\int_c^d g(y)\mathrm{d}y \right)$$

We usually write this as a single step:

$$\int_c^d \int_a^b f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b f(x)\,\mathrm{d}x \int_c^d g(y)\mathrm{d}y$$

C. PHYSICAL INTERPRETATION

The mathematical description of double integrals is defined as a **volume** under a graph (in analogy to the single definite integral being defined as an **area** under a graph), however the physical meaning is often different to a volume, in the same way that single integrals are often not an area.

Remember: Examples of physical interpretation of single integrals:

The single integral can be interpreted as the sum of some quantity (typically some kind of linear density) over a line. In the context of physics, it is always very useful to think about the magnitude/units of the integrand and the result. Don't forget that in addition to the units of the integrand, you also **must consider the units of the differentials**.

Total mass of a rod:

$$\underbrace{M}_{[\mathrm{kg}]} = \int_a^b \underbrace{\lambda(x)}_{[\mathrm{kg/m}]} \underbrace{\mathrm{d}x}_{[m]}$$

Total length of a line segment (trivial):

$$\underbrace{L}_{[\mathrm{m}]} = \int_a^b 1 \underbrace{\mathrm{d}x}_{[m]} = (b-a)$$

Total distance moved during a time interval, knowing the speed:

$$\underbrace{d}_{[\mathrm{m}]} = \int_a^b \underbrace{v(t)}_{[\mathrm{m/s}]} \underbrace{\mathrm{d}t}_{[s]}$$

Total displacement (vector) moved during a time interval, knowing the velocity (vector):

$$\underbrace{\mathbf{s}}_{[\mathrm{m}]} = \int_a^b \underbrace{\mathbf{v}(t)}_{[\mathrm{m/s}]} \underbrace{\mathrm{d}t}_{[s]}$$

Total force (vector) acting on a rod (as a function of a "force density" $\mathbf{f}(x)[N/m]$):

$$\underbrace{\mathbf{F}}_{[\mathrm{N}]} = \int_a^b \underbrace{\mathbf{f}(x)}_{[\mathrm{N/m}]} \underbrace{\mathrm{d}x}_{[m]}$$

Total electric field created by a linear density of charge:

$$\underbrace{\mathbf{E}}_{\substack{[N/c \\ =V/m\,]}} = \int_a^b \underbrace{k_e}_{[\mathrm{N\,m^2\,C^{-2}}]} \underbrace{\frac{\hat{\mathbf{e}}_r}{|\mathbf{r}|^2}}_{[\mathrm{m^{-2}}]} \overbrace{\underbrace{\lambda(x)}_{[\mathrm{C/m}]} \underbrace{\mathrm{dx}}_{[m]}}^{\mathrm{d}q\ [\mathrm{C}]}$$

Examples of physical interpretation of double integrals:

Similarly, the double integral can be interpreted as the addition of some quantity (typically some kind of area density) over an area (or over any two-dimensional parameter space). This is best understood through examples. Again, the units can help us greatly.

Total mass of a planar object covering an area A (with varying surface density $\sigma(x,y)$):

$$\underbrace{M}_{[\mathrm{kg}]} = \iint_A \underbrace{\sigma(x,y)}_{[\mathrm{kg/m^2}]} \underbrace{\mathrm{d}A}_{[\mathrm{m^2}]}$$

Total surface area of an area A (trivial for rectangular areas but useful otherwise!)

$$\underbrace{S}_{[\mathrm{m^2}]} = \iint_A 1 \underbrace{\mathrm{d}A}_{[\mathrm{m^2}]}$$

Total force acting on a wall covering an area A experiencing a non-uniform pressure $P(x,y)$

$$\underbrace{\mathbf{F}}_{[\mathrm{N}]} = \iint_A \underbrace{P(x,y)}_{[\mathrm{N/m^2}]} \underbrace{\mathrm{d}A}_{[\mathrm{m^2}]} \hat{\mathbf{n}}$$

Total volume above sea level of an island with topography height $h(x,y)$ with $h=0$ being sea level:

$$\underbrace{V}_{[\mathrm{m^3}]} = \iint_A \underbrace{h(x,y)}_{[\mathrm{m}]} \underbrace{\mathrm{d}A}_{[\mathrm{m^2}]}$$

Total electric field created by a planar object covering an area A with surface density of charge:

$$\underbrace{\mathbf{E}}_{\substack{[N/c \\ =V/m\,]}} = \iint_A \underbrace{k_e}_{[\mathrm{N\,m^2\,C^{-2}}]} \underbrace{\frac{\hat{\mathbf{e}}_r}{|\mathbf{r}|^2}}_{[\mathrm{m^{-2}}]} \overbrace{\underbrace{\sigma(x,y)}_{[C/m^2]} \underbrace{\underbrace{\mathrm{d}x}_{[\mathrm{m}]} \underbrace{\mathrm{d}y}_{[\mathrm{m}]}}_{\mathrm{d}A\,[\mathrm{m^2}]}}^{\mathrm{d}q\,[\mathrm{C}]}$$

Total price of a plot of land A whose price per square meter is variable $p(x,y)$

$$\underbrace{P}_{[€]} = \iint_A \underbrace{p(x,y)}_{[€/\mathrm{m^2}]} \underbrace{\mathrm{d}A}_{[\mathrm{m^2}]}$$

Average temperature of a surface with non-uniform temperature distribution $T(x,y)$:

$$\underbrace{T_{\mathrm{av}}}_{[\mathrm{K}]} = \frac{\iint_A T(x,y)\,\mathrm{d}A}{\iint_A 1\,\mathrm{d}A} = \frac{1}{S\,[m^2]} \iint_A \underbrace{T(x,y)}_{[\mathrm{K}]} \underbrace{\mathrm{d}A}_{[\mathrm{m^2}]}$$

Average of any quantity $f(x,y)$ over a surface A (also **weighted average** with weight $w(x,y)$):

$$f_{\mathrm{av}} = \frac{\iint_A f(x,y)\,\mathrm{d}A}{\iint_A 1\,\mathrm{d}A} \qquad f_{\mathrm{av}}^{\mathrm{weighted}} = \frac{\iint_A f(x,y)\,w(x,y)\,\mathrm{d}A}{\iint_A w(x,y)\,\mathrm{d}A}$$

Centre of mass (average position weighted by the density) of a planar material:

$$\mathbf{r}_{\mathrm{av}} = \frac{\iint_A \mathbf{r}\,\sigma(x,y)\,\mathrm{d}A}{\iint_A \sigma(x,y)\,\mathrm{d}A}$$

D. TRIPLE (VOLUME) INTEGRALS

Double integrals were justified by problems based on areas and plane 2D objects. Similarly, triple integrals appear in problems related to volumes of 3D objects. These are defined as a limit of an integral sum:

$$I = \sum f(x_i, y_i, z_i)\Delta V_i$$

TRIPLE INTEGRALS IN RECTANGULAR REGION BY ITERATED INTEGRATION

$$\iiint_V f(x,y,z)\,\mathrm{d}V = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)\,\mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z$$

Examples of physical interpretation of triple integrals:

Similarly to the double integral, the triple integral can be interpreted as the addition of some quantity (typically some kind of volumetric density) over a volume (or over any three-dimensional parameter space). This is best understood trough examples. Again, the units can help us greatly.

Total mass of an object occupying a volume V (with varying density $\rho(x,y,z)$):

$$\underbrace{M}_{[\mathrm{kg}]} = \iiint_V \underbrace{\rho(x,y,z)}_{[\mathrm{kg/m^3}]}\ \underbrace{\mathrm{d}V}_{[\mathrm{m^3}]}$$

Total volume of a region Ω

$$\underbrace{V}_{[\mathrm{m^3}]} = \iiint_\Omega 1\ \underbrace{\mathrm{d}V}_{[\mathrm{m^3}]}$$

Total electric field created by a 3D object covering a volume V with varying charge density ρ:

$$\underbrace{\mathbf{E}}_{\substack{[N/c \\ =V/m\,]}} = \iiint_V \underbrace{k_e}_{[\mathrm{N\,m^2\,C^{-2}}]} \underbrace{\frac{\hat{\mathbf{e}}_r}{|\mathbf{r}|^2}}_{[\mathrm{m^{-2}}]} \overbrace{\underbrace{\rho(x,y,z)}_{[C/m^3]}\ \underbrace{\underbrace{\mathrm{d}x}_{[\mathrm{m}]}\ \underbrace{\mathrm{d}y}_{[\mathrm{m}]}\ \underbrace{\mathrm{d}z}_{[\mathrm{m}]}}_{\mathrm{dV}\,[\mathrm{m^3}]}}^{\mathrm{d}q\,[\mathrm{C}]}$$

Average of any quantity $f(x,y,z)$ over a volume V (also **weighted average** with weight $w(x,y,z)$):

$$f_{\mathrm{av}} = \frac{\iiint_V f(x,y,z)\mathrm{d}V}{\iiint_V 1\,\mathrm{d}V} \qquad f_{\mathrm{av}}^{\mathrm{weighted}} = \frac{\iiint_V f(x,y,z)\,w(x,y,z)\mathrm{d}V}{\iiint_V w(x,y,z)\,\mathrm{d}V}$$

Example: center of mass (average position weighted by the density) of a 3D object:

$$\mathbf{r}_{\mathrm{av}} = \frac{\iiint_V \mathbf{r}\,\rho(x,y,z)\,\mathrm{d}V}{\iiint_V \rho(x,y,z)\,\mathrm{d}V}$$

2) a) Calculate the total mass of a cube of side length 2 centred in the origin whose mass density is given by $\rho_m = (1 + xy^2z^2)$ kg/m³

The total mass is given by:

$$\underbrace{M_{\text{tot}}}_{[\text{kg}]} = \iiint_V \underbrace{\rho(x,y,z)}_{[\text{kg/m}^3]} \underbrace{\mathrm{d}V}_{[\text{m}^3]}$$

$$\iiint_V (1 + xy^2z^2)\,\mathrm{d}V = \int_{-1}^{1}\left(\int_{-1}^{1}\left(\int_{-1}^{1}(1 + xy^2z^2)\,\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z$$

We could do the iterated integrals directly, or we can take shortcuts. First, apply linearity (which also works for multiple integrals):

$$\iiint_V (1 + xy^2z^2)\,\mathrm{d}V = \iiint_V 1\,\mathrm{d}V + \iiint_V xy^2z^2\,\mathrm{d}V$$

And then apply separation of each integral (as it is made up of a product of functions of single variables):

$$\begin{aligned}
&= \int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}\mathrm{d}z + \int_{-1}^{1}x\,\mathrm{d}x\int_{-1}^{1}y^2\,\mathrm{d}y\int_{-1}^{1}z^2\,\mathrm{d}z \\
&= (x)_{x=-1}^{x=1}(y)_{y=-1}^{y=1}(z)_{z=-1}^{z=1} + \left(\frac{x^2}{2}\right)_{x=-1}^{x=1}\left(\frac{y^3}{3}\right)_{y=-1}^{y=1}\left(\frac{z^3}{3}\right)_{z=-1}^{z=1} \\
&= (1+1)(1+1)(1+1) + \underbrace{\left(\frac{1}{2}-\frac{1}{2}\right)}_{0}\left(\frac{1}{3}+\frac{1}{3}\right)\left(\frac{1}{3}+\frac{1}{3}\right) \\
&= 8 + 0 = 8\text{ kg}
\end{aligned}$$

Notice how the x term cancelled out the contribution from the variation in mass, so that only the constant density became important. That is because the variation with x had an ODD symmetry, and so all the mass lacking in one side $(x < 0)$ due to the decrease in density, was exactly compensated by the excess mass on the other side $(x > 0)$.

b) Calculate the centre of mass of the cube

The centre of mass is given by:

$$\mathbf{r}_{\text{av}} = \frac{\iiint_V \mathbf{r}\,\rho(x,y,z)\,\mathrm{d}V}{\iiint_V \rho(x,y,z)\,\mathrm{d}V} = \frac{\iiint_V \mathbf{r}\,\rho(x,y,z)\,\mathrm{d}V}{M_{\text{tot}}}$$

We already have the denominator, which is the total mass, so we need to calculate the numerator. The integrand is a **vector**, so the result will be a **vector** too. We need to integrate each component separately:

$$M_{\text{tot}}\mathbf{r}_{\text{av}} = \iiint_V \mathbf{r}\,\rho(x,y,z)\,\mathrm{d}V$$

$$M_{\text{tot}}\begin{pmatrix}x_{\text{av}}\\ y_{\text{av}}\\ z_{\text{av}}\end{pmatrix} = \iiint_V \begin{pmatrix}x\\ y\\ z\end{pmatrix}\rho(x,y,z)\,\mathrm{d}V$$

So, component by component:

$$M_{\mathrm{tot}}x_{\mathrm{av}} = \iiint_V x(1 + xy^2z^2)\,\mathrm{d}V = \iiint_V x\,\mathrm{d}V + \iiint_V x^2y^2z^2\,\mathrm{d}V$$
$$= \underbrace{\int_{-1}^{1} x\,\mathrm{d}x}_{0} \int_{-1}^{1} \mathrm{d}y \int_{-1}^{1} \mathrm{d}z + \underbrace{\int_{-1}^{1} x^2\,\mathrm{d}x}_{\frac{2}{3}} \underbrace{\int_{-1}^{1} y^2\,\mathrm{d}y}_{\frac{2}{3}} \underbrace{\int_{-1}^{1} z^2\,\mathrm{d}z}_{\frac{2}{3}} = 0 + \frac{8}{27}\ \mathrm{kg\cdot m}$$

$$M_{\mathrm{tot}}y_{\mathrm{av}} = \iiint_V y(1 + xy^2z^2)\,\mathrm{d}V = \iiint_V y\,\mathrm{d}V + \iiint_V xy^3z^2\,\mathrm{d}V$$
$$= \int_{-1}^{1} \mathrm{d}x \underbrace{\int_{-1}^{1} y\,\mathrm{d}y}_{0} \int_{-1}^{1} \mathrm{d}z + \underbrace{\int_{-1}^{1} x\,\mathrm{d}x}_{0} \underbrace{\int_{-1}^{1} y^3\,\mathrm{d}y}_{0} \underbrace{\int_{-1}^{1} z^2\,\mathrm{d}z}_{\frac{2}{3}} = 0\ \mathrm{kg\cdot m}$$

$$M_{\mathrm{tot}}z_{\mathrm{av}} = \iiint_V z(1 + xy^2z^2)\,\mathrm{d}V = \iiint_V z\,\mathrm{d}V + \iiint_V xy^2z^3\,\mathrm{d}V$$
$$= \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}y \underbrace{\int_{-1}^{1} z\,\mathrm{d}z}_{0} + \underbrace{\int_{-1}^{1} x\,\mathrm{d}x}_{0} \underbrace{\int_{-1}^{1} y^2\,\mathrm{d}y}_{\frac{2}{3}} \underbrace{\int_{-1}^{1} z^3\,\mathrm{d}z}_{0} = 0\ \mathrm{kg\cdot m}$$

So, overall, dividing the three results by $M_{\mathrm{tot}} = 8\ \mathrm{kg}$ we get the centre of mass (weighted average position):

$$\mathbf{r}_{\mathrm{av}} = \begin{pmatrix} x_{\mathrm{av}} \\ y_{\mathrm{av}} \\ z_{\mathrm{av}} \end{pmatrix} = \begin{pmatrix} 1/27 \\ 0 \\ 0 \end{pmatrix}\ \mathrm{m}$$

Notice how, this time, the uniform density term (1) always integrated to zero for the three integrals, because obviously, the centre of mass of a uniform density cube centred at the origin is the origin itself. However, the variation in density term (xy^2z^2) does provide a non-zero centre of mass caused by the odd symmetry of the density along x, which moves the centre of mass in the x direction, as the $x = 1$ side of the cube is denser than the $x = -1$ side. The even symmetry of the density along y and z means that the center of mass is not moved in the y nor z directions.

PROBLEMS

3) Calculate the following integrals over the rectangular region $R: 0 \leq x \leq 2$ and $1 \leq y \leq 2$

$$I_1 = \iint_R (x+y)\,\mathrm{d}A$$

$$I_2 = \iint_R x\,\mathrm{d}A$$

$$I_3 = \iint_R 4xye^{-x^2-y^2}\,\mathrm{dA}$$

Solution:

$$I_1 = \iint_R (x+y)\,\mathrm{d}A = \int_0^2 \int_1^2 (x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=2} \mathrm{d}x = \int_0^2 \left(2x + 2 - x - \frac{1}{2}\right)\mathrm{d}x$$
$$= \int_0^2 \left(x + \frac{3}{2}\right)\mathrm{d}x = \left[\frac{1}{2}x^2 + \frac{3}{2}x\right]_{x=0}^{x=2} = 2 + 3 = 5$$

$$I_2 = \iint_R (x)\,\mathrm{d}A = \int_0^2 \int_1^2 (x)\,\mathrm{d}y\,\mathrm{d}x = (\text{separation}) = \int_1^2 \mathrm{d}y \int_0^2 x\,\mathrm{d}x = (1)\left(\frac{1}{2}x^2\right)_{x=0}^{x=2} = 2$$

$$I_3 = \iint_R \left(4xye^{-x^2-y^2}\right)\mathrm{d}A = \int_0^2 \int_1^2 \left(2xe^{-x^2}\,2ye^{-y^2}\right)\mathrm{d}y\,\mathrm{d}x = (\text{separation})$$
$$= \int_1^2 \left(2ye^{-y^2}\right)\mathrm{d}y \int_0^2 \left(2xe^{-x^2}\right)\mathrm{d}x = \left[-e^{-y^2}\right]_{y=1}^{y=2}\left[-e^{-x^2}\right]_{x=0}^{x=2}$$
$$= (-e^{-4} + e^{-1})(-e^{-4} + e^{0}) = e^{-8} - e^{-5} - e^{-4} + e^{-1}$$

4) Calculate the following triple integral over the region V given by the unit cube (side lengths one) whose centre is at the origin

$$I = \iiint_V \sin(x)\ \mathrm{d}V$$

Solution:

$$I = \iiint_V \sin(x)\ \mathrm{d}V = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\sin(x))\,\mathrm{d}x \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}y \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}z$$
$$= [-\cos x]_{x=-\frac{1}{2}}^{x=\frac{1}{2}} [y]_{-\frac{1}{2}}^{\frac{1}{2}} [z]_{-\frac{1}{2}}^{\frac{1}{2}} = (0)(1)(1) = 0$$

5) Calculate the centre of mass for a square plate $0 \leq x \leq 1$ and $0 \leq y \leq 1$ whose mass density is $\sigma = ye^x$.

Remember:

$$\mathbf{r}_{\text{av}} = \frac{\iint_A \mathbf{r}\, \sigma(x,y)\, \mathrm{d}A}{\iint_A \sigma(x,y)\, \mathrm{d}A}$$

Solution:

Let's calculate the denominator first (the total mass)

$$\iint_A \sigma(x,y)\, \mathrm{d}A = \iint_A ye^x\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 e^x\, \mathrm{d}x \int_0^1 y\, \mathrm{d}y = [e^x]_{x=0}^{x=1} \left[\frac{1}{2}y^2\right]_{y=0}^{y=1} = \frac{e-1}{2}$$

Now calculate the three components of the nominator (integrate component by component, knowing that $\mathbf{r} = (x, y, 0)$**:**

$$\iint_A \mathbf{r}\, \sigma(x,y)\, \mathrm{d}A = \begin{pmatrix} \iint_A x\, \sigma(x,y)\, \mathrm{d}A \\ \iint_A y\, \sigma(x,y)\, \mathrm{d}A \end{pmatrix}$$

The x-component:

$$\iint_A x\, \sigma(x,y)\, \mathrm{d}A = \iint_A xye^x\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 xe^x\, \mathrm{d}x \int_0^1 y\, \mathrm{d}y$$

$\int_0^1 xe^x\, \mathrm{d}x$ can be done using integration by parts $\int uv' = uv - \int vu'$

Take $\begin{cases} u = x \rightarrow u' = 1 \\ v' = e^x \rightarrow v = e^x \end{cases}$

$$\int_0^1 xe^x\, \mathrm{d}x = [xe^x]_0^1 - \int_0^1 e^x\, \mathrm{d}x = (e - 0) - [e^x]_0^1 = e - e + 1 = 1$$

$$= \int_0^1 xe^x\, \mathrm{d}x \int_0^1 y\, \mathrm{d}y = (1)\left[\frac{1}{2}y^2\right]_0^1 = \frac{1}{2}$$

The y-component:

$$\iint_A y\, \sigma(x,y)\, \mathrm{d}A = \iint_A y^2 e^x\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 e^x\, \mathrm{d}x \int_0^1 y^2\, \mathrm{d}y = [e^x]_0^1 \left[\frac{1}{3}y^3\right]_0^1 = \frac{e-1}{3}$$

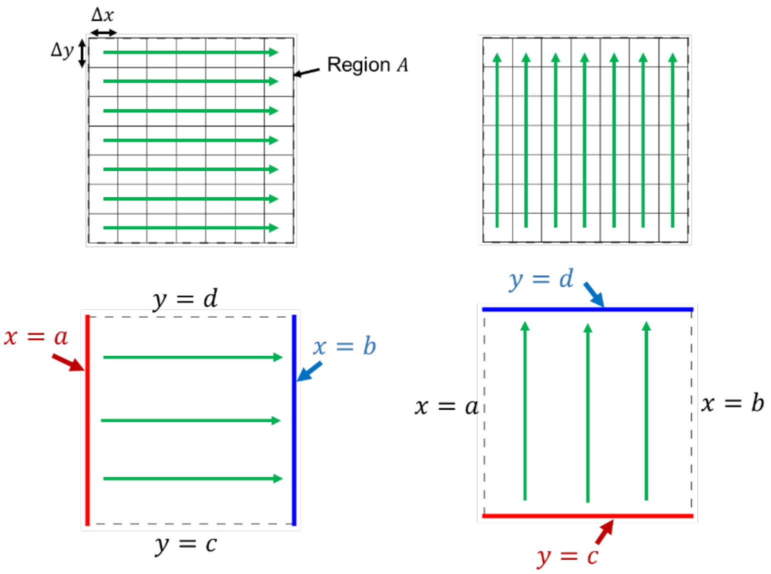
So, putting it all together:

$$\mathbf{r}_{\text{av}} = \frac{\left(\frac{1}{2}, \frac{e-1}{3}\right)}{\frac{e-1}{2}} = \left(\frac{1}{e-1}, \frac{2}{3}\right)$$

4.2 DOUBLE AND TRIPLE INTEGRALS

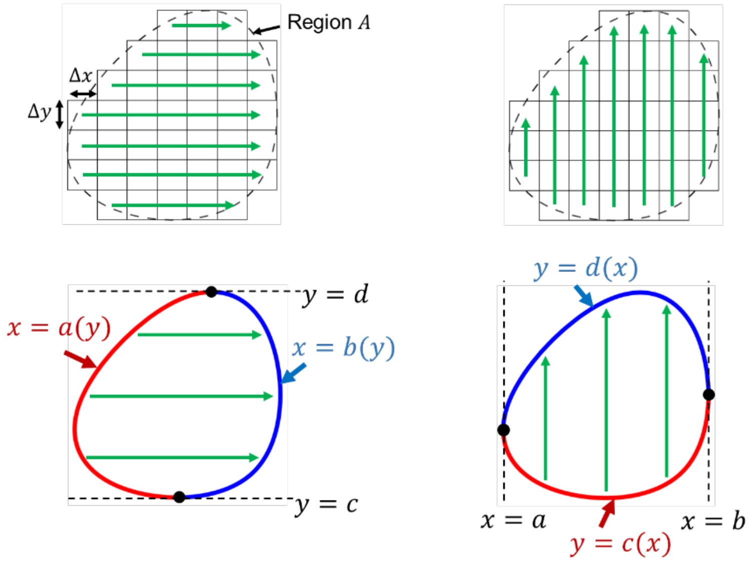
A. DOUBLE INTEGRALS IN NON-RECTANGULAR REGIONS

In a rectangular area, the double integral was performed by integrating along one direction first, with constant limits of integration, and then integrating this function along the other remaining direction:



$$\iint_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x \right) \mathrm{d}y \qquad \text{or} \qquad \int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

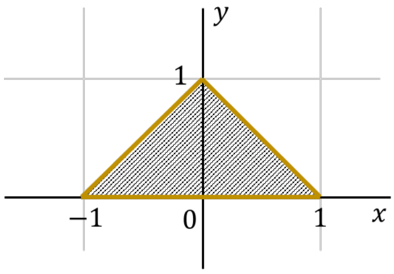
When the region is **not rectangular**, we have to change the limits of integration to be a function:



$$\iint_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y \qquad \text{or} \qquad \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

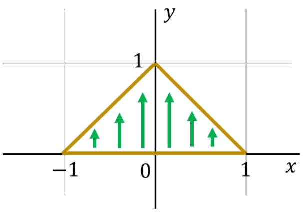
Always remember that **the integration removes the variable of integration, but also adds back any variable which appears in the limits of integration**. Therefore, if we first integrate along x, we want the result to be a function of y only, with no x, ready for the outer integration, so the limits cannot involve x. On the other hand, if we are first integrating along y, then the limits must be functions of x and not involve y. **In principle, the order of integration is an arbitrary choice. In practice the difficulty of the integral (or even the feasibility) can depend on the order we choose.**

1) Calculate the area of the triangle shown by using double integrals



Solution: To calculate area using a double integral, we need to use 1 as the integrand. We will show both possible orders of iterated integration.

1) Doing first the integration along y. This will mean that we must divide the area into two regions.

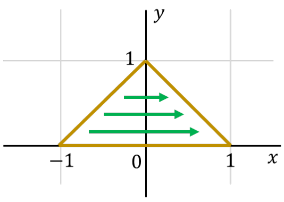


$$A = \iint_T \mathrm{d}x\,\mathrm{d}y = \iint_{T_1} \mathrm{d}x\,\mathrm{d}y + \iint_{T_2} \mathrm{d}x\,\mathrm{d}y = \int_{-1}^{0}\left(\int_0^{1+x} \mathrm{d}y\right)\mathrm{d}x + \int_0^1\left(\int_0^{1-x}\mathrm{d}y\right)\mathrm{d}x$$

$$= \int_{-1}^{0} [y]_{y=0}^{y=1+x}\mathrm{d}x + \int_0^1 [y]_{y=0}^{y=1-x}\mathrm{d}x = \int_{-1}^{0}(1+x)\mathrm{d}x + \int_0^1 (1-x)\mathrm{d}x$$

$$= \left[x + \frac{1}{2}x^2\right]_{x=-1}^{x=0} + \left[x - \frac{1}{2}x^2\right]_{x=0}^{x=1} = 1 - \frac{1}{2} + 1 - \frac{1}{2} = 1$$

2) Doing first the integration along x. Then we can do it in one single integral as we can use a function for both the lower and upper limit of the inner integral.

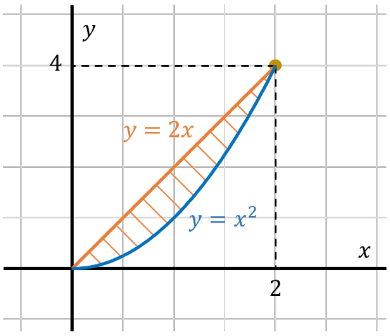


$$A = \iint_T \mathrm{d}x\,\mathrm{d}y = \int_0^1\left(\int_{y-1}^{1-y}\mathrm{d}x\right)\mathrm{d}y = \int_0^1 [x]_{x=y-1}^{x=1-y}\mathrm{d}y = \int_0^1 (2-2y)\mathrm{d}y = [2y - y^2]_{y=0}^{y=1} = 1$$

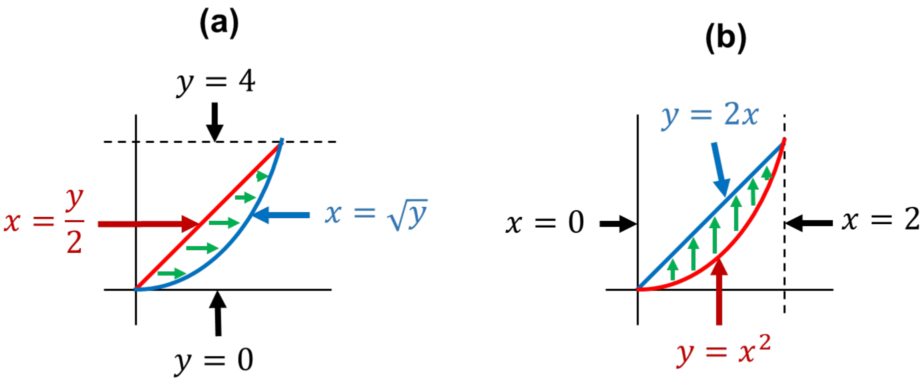
As you can see, this second choice made things easier. Interestingly, in some cases, one path leads to an impossible integral while the other path is simple.

2) Problem: Find $\iint_R (4x+2)\,\mathrm{d}A$ where R is the region of the plane contained between the graphs $y = x^2$ and $y = 2x$.

Solution: First, sketch the region and label the curves. The two graphs meet at $x = 0$ and $x = 2$ (the two simultaneous solutions).



We can choose two different orders of integration:



(a) Integrating first along x:

$$\iint_R (4x+2)\,\mathrm{d}A = \int_0^4 \left(\int_{y/2}^{\sqrt{y}} (4x+2)\mathrm{d}x \right) \mathrm{d}y = \int_0^4 (2x^2+2x)_{x=y/2}^{x=\sqrt{y}} \mathrm{d}y$$
$$= \int_0^4 \left(2y + 2\sqrt{y} - \frac{1}{2}y^2 - y \right) \mathrm{d}y = \int_0^4 \left(-\frac{1}{2}y^2 + y + 2y^{1/2} \right) \mathrm{d}y$$
$$= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + \frac{4}{3}y^{3/2} \right)_{y=0}^{y=4} = -\frac{64}{6} + \frac{16}{2} + \frac{4}{3}8 = -\frac{32}{3} + 8 + \frac{32}{3} = 8$$

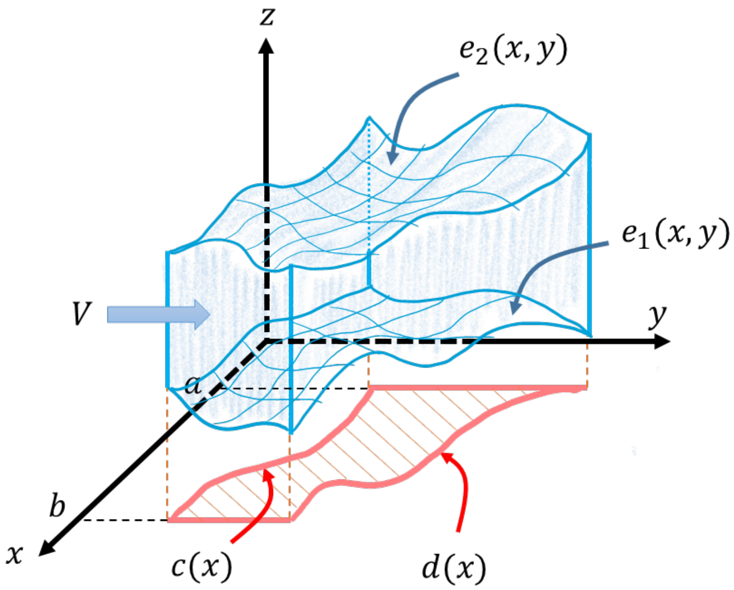
(b) Integrating first along y:

$$\iint_R (4x+2)\,\mathrm{d}A = \int_0^2 \left(\int_{x^2}^{2x} (4x+2)\mathrm{d}y \right) \mathrm{d}x = \int_0^2 (4xy+2y)_{y=x^2}^{y=2x} \mathrm{d}x$$
$$= \int_0^2 (8x^2 + 4x - 4x^3 - 2x^2)\,\mathrm{d}x = \int_0^2 (6x^2 - 4x^3 + 4x)\,\mathrm{d}x$$
$$= (2x^3 - x^4 + 2x^2)_{x=0}^{x=2} = 16 - 16 + 8 = 8$$

Both give the same result.

B. TRIPLE INTEGRALS IN NON-RECTANGULAR REGIONS

We can define the limits of each successive integration as functions which depend only on the variables whose integration we have not yet done.



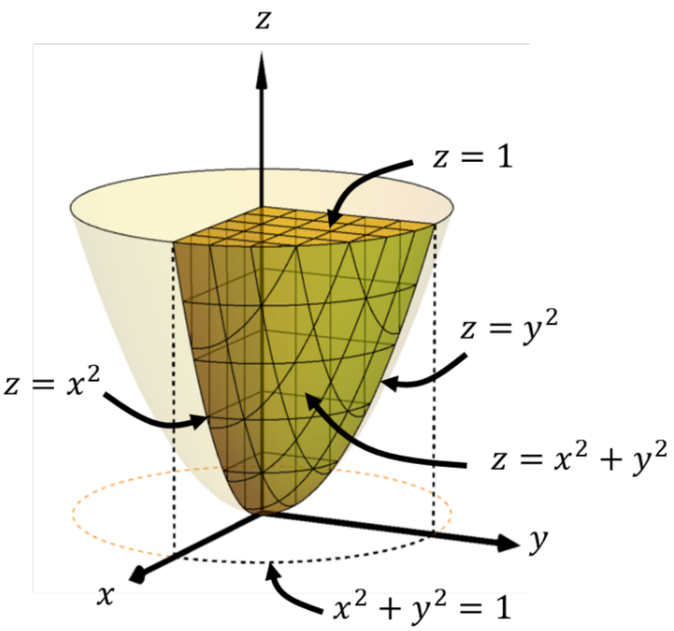
$$\iiint_V f(x,y,z)\,\mathrm{d}V = \int_a^b \underbrace{\left[\int_{c(x)}^{d(x)} \underbrace{\left(\int_{e_1(x,y)}^{e_2(x,y)} f(x,y,z)\,\mathrm{d}z\right)}_{g(x,y)} \mathrm{d}y\right]}_{h(z)} \mathrm{d}x$$

As usual, the order of integration is not fixed (but can be important for ease of calculation).

The procedure is **exactly** equal to projecting the whole volume into a plane (e.g. the XY plane) and then performing a double integral of function $g(x,y)$ over the projected area.

3) Calculate the integral of $f(x,y,z) = x$ over the region given by $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq z \leq 1$.

Solution:



We have many possibilities on the order of integration, some will be harder than others:

- Start integration along z (i.e. projecting onto the XZ plane)

$$I = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\int_{x^2+y^2}^{1} x \, \mathrm{d}z \right) \mathrm{d}y \right] \mathrm{d}x$$

- Start integration along x (i.e. projecting along the YZ plane) and then integrating along z first:

$$I = \int_0^1 \left[\int_{y^2}^{1} \left(\int_0^{\sqrt{z-y^2}} x \, \mathrm{d}x \right) \mathrm{d}z \right] \mathrm{d}y$$

- Start integration along x (i.e. projecting along the YZ plane) and then integrating along y first:

$$I = \int_0^1 \left[\int_0^{\sqrt{z}} \left(\int_0^{\sqrt{z-y^2}} x \, \mathrm{d}x \right) \mathrm{d}y \right] \mathrm{d}z$$

Let's choose the third route:

$$I = \int_0^1 \left[\int_0^{\sqrt{z}} \left(\int_0^{\sqrt{z-y^2}} x \, \mathrm{d}x \right) \mathrm{d}y \right] \mathrm{d}z = \int_0^1 \left[\int_0^{\sqrt{z}} \left(\frac{x^2}{2} \right)_{x=0}^{x=\sqrt{z-y^2}} \mathrm{d}y \right] \mathrm{d}z = \frac{1}{2} \int_0^1 \left[\int_0^{\sqrt{z}} (z - y^2) \, \mathrm{d}y \right] \mathrm{d}z$$

$$= \frac{1}{2} \int_0^1 \left[zy - \frac{1}{3} y^3 \right]_{y=0}^{y=\sqrt{z}} \mathrm{d}z = \frac{1}{2} \int_0^1 \left(z^{3/2} - \frac{1}{3} z^{3/2} \right) \mathrm{d}z = \frac{1}{3} \int_0^1 z^{3/2} \, \mathrm{d}z = \frac{1}{3} \left[\frac{2}{5} z^{\frac{5}{2}} \right]_{z=0}^{z=1}$$

$$= \frac{2}{15}$$

C. CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

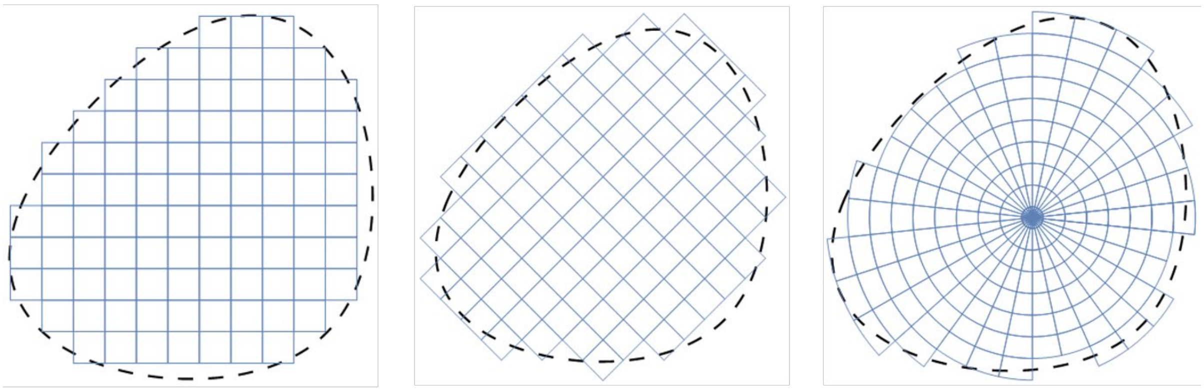
In many cases it is possible to simplify calculation of a double integral (or calculate it at all!) by introducing new variables u and v via connection equations:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

This means that, in the definition of the integral, instead of "chopping" the integration area into rectangles bounded by lines of constant x and constant y, we "chop it" by using curves of constant u and constant v.

For example, we can change the integration to coordinates that are rotated, or to polar coordinates:

$$\begin{cases} u = x \\ v = y \end{cases} \qquad \begin{cases} u = (x+y)/\sqrt{2} \\ v = (x-y)/\sqrt{2} \end{cases} \qquad \begin{cases} u = \sqrt{x^2+y^2} \\ v = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$



$$\begin{cases} x = u \\ y = v \end{cases} \qquad \begin{cases} x = (u+v)/\sqrt{2} \\ y = (u-v)/\sqrt{2} \end{cases} \qquad \begin{cases} x = u\cos v \\ y = u\sin v \end{cases}$$

When doing a change of variables we need to:

(1) Change the integrand to the new variables: $f(x,y) \to f\big(x(u,v), y(u,v)\big) = f'(u,v)$.

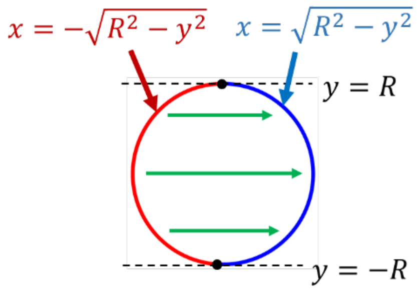
(2) Change the limits to make sure they define the same integration region in the new coordinates. The limits may look very different in the new (u,v) coordinates, usually simpler if the coordinates are chosen wisely, and may even allow the calculation of the integral.

(3) Change the element of area $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ into some function $\mathrm{d}A = j(u,v)\,\mathrm{d}u\,\mathrm{d}v$ that accounts for the area of the new "chopping method". We will see that $j(u,v) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

Let's explain this through an example:

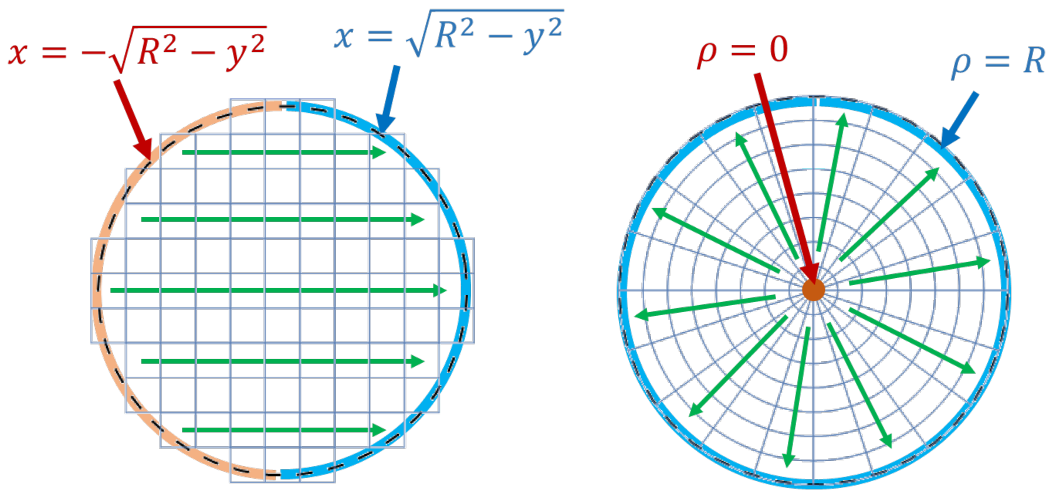
4) Calculate the area of a circle by doing a double integral (with an integrand of 1).

The limits of integration in rectangular coordinates would require us to integrate:



$$A_{circ} = \iint_C \mathrm{dA} = \iint_C \mathrm{d}x\,\mathrm{d}y = \int_{-R}^{R}\left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \mathrm{d}x\right)\mathrm{d}y$$

Which is a relatively ugly integral with square roots! However, we can simplify this calculation by using polar coordinates; then the limits of integration become simple $\rho \in [0, R]$ and $\phi \in [0,2\pi]$.



How <u>NOT</u> to do it: Do steps (1) and (2) but forget (3) changing the differential of area

$$A_{circ} = \iint_C \mathrm{d}A \overset{\text{WRONG}}{\neq} \iint_C \mathrm{d}\phi\,\mathrm{d}\rho = \int_0^R\left(\int_0^{2\pi}\mathrm{d}\phi\right)\mathrm{d}\rho = 2\pi R \quad (\text{WRONG})$$

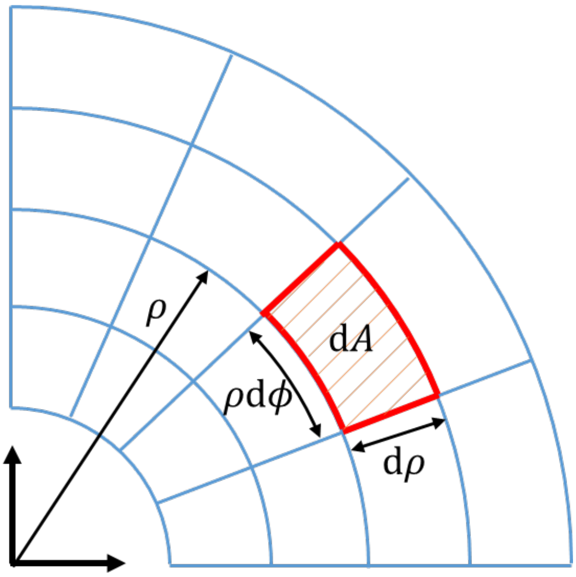
Above, we simply changed the limits of integration to the ones for polar coordinates.

However, by doing this, we obtained the area of a **rectangle** of sides 2π and R. That rectangle is exactly how the region of integration of a circle looks like when viewed in polar coordinates, however, how is mathematics going to know that we meant to be using polar coordinates and not integrating a rectangle? How can maths distinguish $\int_0^R\left(\int_0^{2\pi}\mathrm{d}\phi\right)\mathrm{d}\rho$ from $\int_0^R\left(\int_0^{2\pi}\mathrm{d}x\right)\mathrm{d}y$? The answer is it can't. Maths doesn't care about the symbols we use. There must be something that tells mathematics that we are working in deformed coordinates, and the answer is that we need to use $\mathrm{d}A = \rho\,\mathrm{d}\phi\,\mathrm{d}\rho$. This is because the differentials of area that are closer to the origin, are clearly more "squashed" and "count" less towards the area than those far away!

So how do we know what to write for $\mathrm{d}A$ when changing variables? There are two methods:

FIND $\mathbf{d}A$ BY GEOMETRICAL INTUITION:

(Only works for simple cases… which is most of the times). Picture the differential area by considering it enclosed between lines of constant u and v, and then find out an expression for the area created, in terms of $\mathrm{d}u$ and $\mathrm{d}v$. For example, for polar coordinates, consider a small increase in radius $\mathrm{d}\rho$ and a small increase in angle $\mathrm{d}\phi$:



Since the increase in ρ is infinitesimal, we can assume that the length of the arcs $\rho\ \mathrm{d}\phi$ and $(\rho + \mathrm{d}\rho)d\phi$ are equal (by ignoring squared differentials). Also, the increase in angle ϕ is infinitesimal, so the curved arc becomes a straight line. **<u>In the limit, the area tends to a parallelogram</u>** (in this case a rectangle) whose area is exactly the product of the sides $dA = \rho\ \mathrm{d}\rho\ \mathrm{d}\phi$.

Let's redo our calculation of the area of a circle now:

$$A_{circ} = \iint_C \mathrm{d}A \overset{\text{CORRECT}}{=} \iint_C \rho\ \mathrm{d}\phi\ \mathrm{d}\rho$$

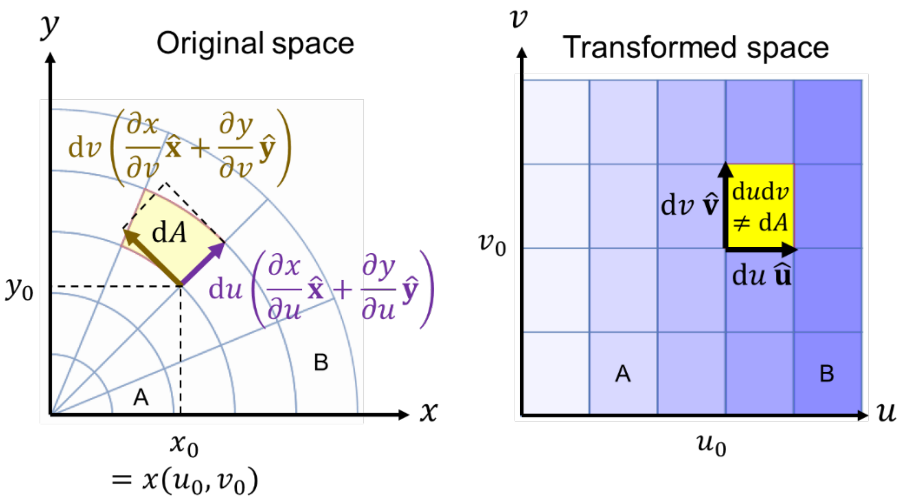
The area of the circle in polar coordinates becomes simple limits of integration $\rho \in [0, R]$ and $\phi \in [0,2\pi]$, so we can substitute:

$$= \int_0^{2\pi} \left(\int_0^R \rho\ \mathrm{d}\rho \right) \mathrm{d}\phi = \int_0^{2\pi} \mathrm{d}\phi \int_0^R \rho\ \mathrm{d}\rho = [\phi]_{\phi=0}^{\phi=2\pi} \cdot \left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=R} = \pi R^2$$

giving us the correct answer for the area of a circle.

FIND dA USING THE JACOBIAN - GENERAL RECIPE FOR CHANGE OF VARIABLES

Consider what you do when you integrate a function of x and y over some region. Basically, you chop up the region into boxes of area $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, evaluate the function at a point in each box, multiply it by the area of the box, and sum over all boxes. What you do when changing variables is to chop the region into boxes that are not rectangular, and instead chop it along lines that are defined by some function, $u(x, y)$, being constant. Now in order to evaluate the sum of boxes, which may be slanted now, you need to find the "area of box" for the new boxes - it's not $\mathrm{d}A = \mathrm{d}u\,\mathrm{d}v$ anymore.



As the boxes are infinitesimal, the edges cannot be curved, so they must be parallelograms (adjacent lines of constant u or constant v are parallel). The parallelograms are defined by two vectors - the vector resulting from a small change in u, and the one resulting from a small change in v. In component form, these vectors are $\mathrm{d}u\left(\frac{\partial x}{\partial u}\hat{\mathbf{x}} + \frac{\partial y}{\partial u}\hat{\mathbf{y}}\right)$ and $\mathrm{d}v\left(\frac{\partial x}{\partial v}\hat{\mathbf{x}} + \frac{\partial y}{\partial v}\hat{\mathbf{y}}\right)$, where the derivatives are evaluated at x_0 and y_0. To see this, imagine moving a small distance $\mathrm{d}u$ along a line of constant v. What's the change in x when you change u but hold v constant? The partial of x with respect to u, times $\mathrm{d}u$. Same with the change in y. (Notice that this involves writing x and y as functions of u, v, rather than the other way around. The main condition of a change in variables is that both ways are possible). The area of a parallelogram bounded by two vectors is given by the determinant of the matrix formed when the vectors are used as columns:

$$\mathrm{d}A(u,v) = \begin{vmatrix} \mathrm{d}u\,\dfrac{\partial x}{\partial u} & \mathrm{d}v\,\dfrac{\partial x}{\partial v} \\ \mathrm{d}u\,\dfrac{\partial y}{\partial u} & \mathrm{d}v\,\dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u\,\mathrm{d}v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

Which gives us the general recipe for change of variables in multiple integration. We always must take the absolute value of it, as dA should always be positive. The matrix whose determinant we are taking is called the Jacobian of the transformation (see lesson on multivariate derivative) and it can be extended to more dimensions:

$$\mathbf{J} = \frac{\partial(x, y, \dots, z)}{\partial(u, v, \dots, w)} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v & \dots & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \dots & \partial y/\partial w \\ \vdots & \vdots & \ddots & \vdots \\ \partial z/\partial u & \partial z/\partial v & \dots & \partial z/\partial w \end{pmatrix}$$

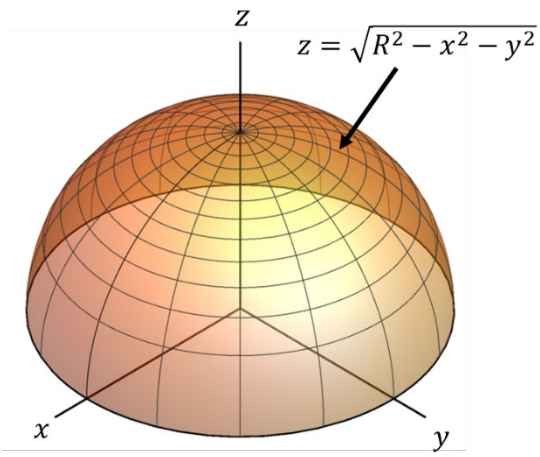
(Parts of text taken from user Dan at StackExchange)

The general rule for a change of variables in double integration is:

$$\iint_{A_{(xy)}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{A_{(uv)}} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

5) Calculate the volume of a hemisphere using a double integral in polar coordinates.

We are going to integrate the function $z=\sqrt{R^2-x^2-y^2}$ over an area in the XY plane given by a disk of radius R, i.e. $\rho \in [0,R]$ and $\phi \in [0,2\pi]$.



$$V=\iint_D z(\rho,\phi)\,\mathrm{d}A = \iint_D \sqrt{R^2-x^2-y^2}\,\mathrm{d}A = \begin{Bmatrix} x=\rho\cos\phi \\ y=\rho\sin\phi \end{Bmatrix} = \iint_D \sqrt{R^2-\rho^2}\,\underbrace{(\rho\,\mathrm{d}\phi\,\mathrm{d}\rho)}_{\mathrm{d}A}$$

$$=\int_0^R\left(\int_0^{2\pi}\rho\sqrt{R^2-\rho^2}\,\mathrm{d}\phi\right)\mathrm{d}\rho = (\text{separation}) = \int_0^{2\pi} d\phi \int_0^R \rho\,(R^2-\rho^2)^{\frac{1}{2}}\,\mathrm{d}\rho$$

$$=(2\pi)\left[\frac{1}{-3}(R^2-\rho^2)^{\frac{3}{2}}\right]_{\rho=0}^{\rho=R} = (2\pi)\left(\frac{1}{3}R^3\right) = \frac{2}{3}\pi R^3$$

which is half the volume of a sphere.

6) Calculate the total mass of a disk of radius R whose density increases with the radius as $\sigma(\rho) = 1+\rho^2$.

If we tried doing it in rectangular coordinates:

$$M=\iint_D \sigma(x,y)\,\mathrm{d}A = \int_{-R}^{R}\left(\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}}(1+x^2+y^2)\;\mathrm{d}x\right)\mathrm{d}y = (\text{long procedure}\ldots)$$

In polar coordinates it becomes very easy:

$$M=\iint_D \sigma(x,y)\,\mathrm{d}A = \iint_D (1+\rho^2)\,\underbrace{(\rho\,\mathrm{d}\rho\,\mathrm{d}\phi)}_{\mathrm{d}A} = \int_0^{2\pi}\left(\int_0^R \rho(1+\rho^2)\;\mathrm{d}\rho\right)\mathrm{d}\phi$$

$$=\int_0^{2\pi}\mathrm{d}\phi\int_0^R(\rho+\rho^3)\,\mathrm{d}\rho = 2\pi\left(\frac{R^2}{2}+\frac{R^4}{4}\right) = \pi R^2\left(1+\frac{1}{2}R^2\right)$$

CHANGE OF VARIABLES IN TRIPLE INTEGRALS

Change of variables in triple integrals are the same as in double integrals. We change the variables, the limits, and the volume element. The Jacobian is now a 3-by-3 matrix:

$$\iiint_{V_{(xyz)}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{V_{(uvw)}} \underbrace{f\big(x(u,v,w),y(u,v,w),z(u,v,w)\big)}_{f'(u,v,w)} \underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}_{\mathrm{d}V}$$

7) Calculate the volume element $\mathrm{d}V$ in spherical coordinates:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

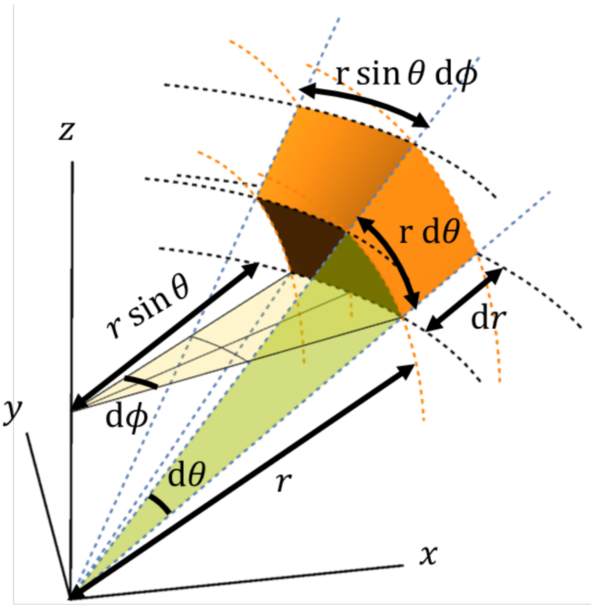
We can do this formally via calculation of the Jacobian determinant:

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial r & \partial z/\partial\phi & \partial z/\partial\theta \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix}$$
$$= \cos\theta\,(-r^2\sin^2\phi\sin\theta\cos\theta - r^2\cos^2\phi\sin\theta\cos\theta)$$
$$- \sin\theta\,(r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi)$$
$$= -r^2\sin\theta\cos^2\theta - r^2\sin^3\theta = -r^2\sin\theta\,(\cos^2\theta + \sin^2\theta) = -r^2\sin\theta$$

Therefore:

$\mathrm{d}V = \left|\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}\right| \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = r^2\sin\theta\;\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ (always take the positive value as it is a volume)

We can also obtain it from geometrical intuition, although it is tricky as one needs to accurately picture where the angles are subtended from (θ is sustained from the origin, but ϕ is sustained from the z-axis):



The volume element for commonly used coordinate systems (given in the exam):

Coordinate system	Volume element $\mathrm{d}V$
Rectangular (x, y, z)	$\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$
Cylindrical (ρ, ϕ, z)	$\rho\ \mathrm{d}\rho\ \mathrm{d}\phi\ \mathrm{d}z$
Spherical (r, θ, ϕ)	$r^2 \sin\theta\ \ \mathrm{d}r\ \mathrm{d}\phi\ \mathrm{d}\theta$

8) Calculate the volume of a sphere using a volume integral in spherical coordinates

In spherical coordinates, a sphere is given by $r \in [0, R]$, $\phi \in [0,2\pi]$ and $\theta \in [0, \pi]$.

Therefore, the volume can be calculated in spherical coordinates as:

$$V = \iiint_S 1\ \mathrm{d}V = \iiint_S \overbrace{(r^2 \sin\theta)\ \mathrm{d}\theta\ \mathrm{d}r\ \mathrm{d}\phi}^{\mathrm{d}V} = \int_0^R \left[\int_0^{2\pi} \left(\int_0^{\pi} r^2 \sin\theta\ \mathrm{d}\theta\right) \mathrm{d}\phi\right] dr$$

$$= \left(\int_0^{2\pi} \mathrm{d}\phi\right)\left(\int_0^R r^2\ \mathrm{d}r\right)\left(\int_0^{\pi} \sin\theta\ \ \mathrm{d}\theta\right) = (2\pi)\left(\frac{R^3}{3}\right)[-\cos\theta]_{\theta=0}^{\theta=\pi} = \frac{4}{3}\pi R^3$$

Is this the first time in your education that you see this equation proven to you, rather than given?

If only Archimedes had known about this method! He spent years trying to figure out the volume of a sphere. He finally did it, in an amazingly clever way by realizing that the volume between a sphere and a cylinder that contains it was, plane by plane, equal to that of a cone. He requested that his tombstone display a sphere inscribed in a cylinder.

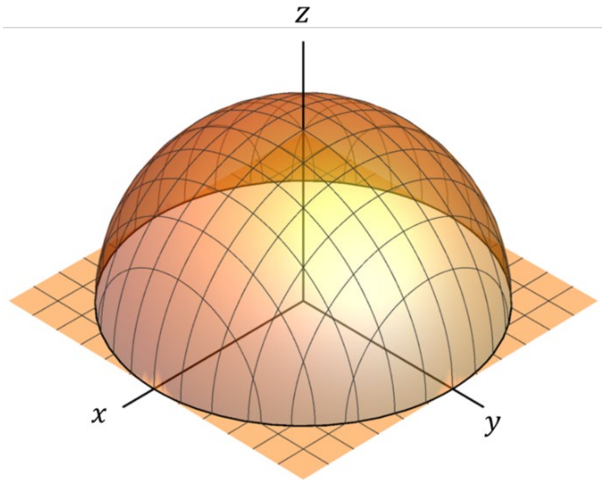
Here we solved it, start-to-finish, on just two lines and requiring only trivial integrals. Beautiful.

Compare the efficiency of this calculation in polar coordinates to what would have been required if we were using rectangular coordinates:

VOLUME OF A SPHERE BY INTEGRATION IN RECTANGULAR COORDINATES.

To illustrate the usefulness of alternative coordinate systems, let's calculate the volume of a sphere in rectangular coordinates (which as we saw can be done in two lines using spherical coordinates).

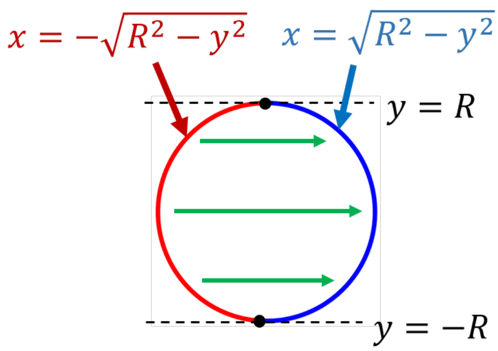
9) Calculate the volume of a hemisphere of radius R centred at zero with $z \geq 0$ using double integration in rectangular coordinates

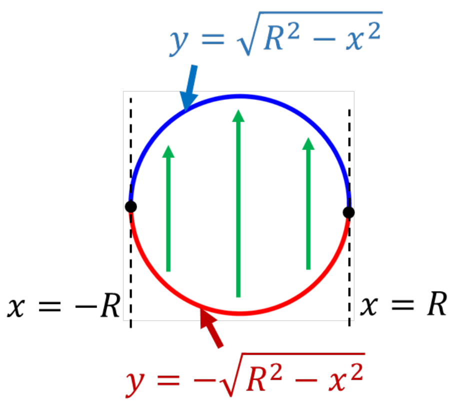


Solution:

$$V = \iint_D z(x, y)\mathrm{d}x\,\mathrm{d}y$$

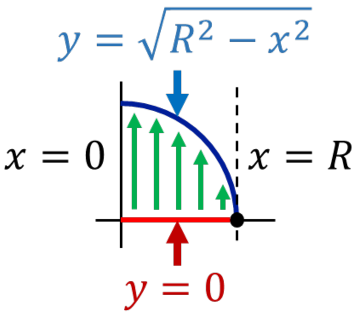
With $z(x, y) = \sqrt{R^2 - x^2 - y^2}$ being the height of the hemisphere, and the region of integration D being the disk centred at 0 with radius R. We first need to find the equations for the limits of integration. Knowing that the equation for the circle is $x^2 + y^2 = R^2$ we can find the limits of integration as a function $y(x) = \pm\sqrt{R^2 - x^2}$ or we could have written it as a function $x(y) = \pm\sqrt{R^2 - y^2}$. Using one or the other would depend on the direction of integration we choose:





$$\int_{-R}^{R}\left(\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \sqrt{R^2 - x^2 - y^2}\ \mathrm{d}x\right)\mathrm{d}y \qquad \int_{-R}^{R}\left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\ \mathrm{d}y\right)\mathrm{d}x$$

Now, we can be smart and simplify our lives by using arguments of symmetry. The volume of the hemisphere will be four times the volume of only one quarter. Therefore;



$$V=4\int_0^R\left(\int_0^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}\,\mathrm{d}y\right)\mathrm{d}x$$

This is a very common integral which appears a lot when dealing with spherical surfaces.

$$\int\sqrt{K^2-x^2}\mathrm{d}x=\frac{1}{2}\left(x\sqrt{K^2-x^2}+K^2\arcsin\left(\frac{x}{K}\right)\right)+c$$

Proof: Change of variables $x=K\sin u\ \rightarrow\ \mathrm{d}x=K\cos u$, which means $\sqrt{K^2-x^2}=\sqrt{K^2-K^2\sin^2 u}=K\cos u$:

$$\int\sqrt{K^2-x^2}\mathrm{d}x=K^2\int\cos^2 u\ \mathrm{d}u=K^2\int\left(\frac{1}{2}+\frac{1}{2}\cos(2u)\right)\mathrm{d}u=K^2\left(\frac{u}{2}+\frac{1}{4}\sin(2u)\right)+c$$

Then apply $\sin(2u)=2\sin u\cos u=2\sin u\ \sqrt{1-\sin^2 u}$

$$=K^2\left(\frac{u}{2}+\frac{1}{2}\sin u\sqrt{1-\sin^2 u}\right)+c$$

and substitute back the change of variables $u=\arcsin(x/K)$ and $\sin u=x/K$

$$=\frac{1}{2}\left(x\sqrt{K^2-x^2}+K^2\arcsin\left(\frac{x}{K}\right)\right)+c$$

I will provide this integral in the exam if it is needed.

Therefore, the double integral can be solved as follows (use $K^2=R^2-x^2$ as constant for the inner integral):

$$V=4\int_0^R\left[\frac{1}{2}\left(y\sqrt{R^2-x^2-y^2}+(R^2-x^2)\arcsin\frac{y}{\sqrt{R^2-x^2}}\right)\right]_{y=0}^{y=\sqrt{R^2-x^2}}\mathrm{d}x$$

$$=4\int_0^R\frac{1}{2}\left(\sqrt{R^2-x^2}\underbrace{\sqrt{R^2-x^2-R^2+x^2}}_{0}+(R^2-x^2)\underbrace{\arcsin 1}_{\pi/2}-0-(R^2-x^2)\underbrace{\arcsin 0}_{0}\right)\mathrm{d}x$$

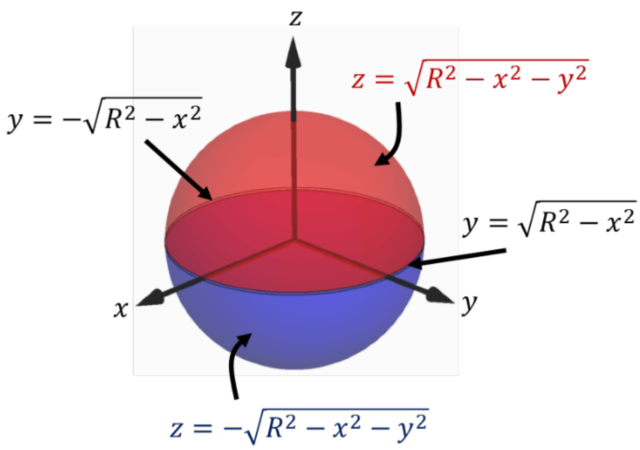
$$=4\int_0^R\frac{1}{2}(R^2-x^2)\frac{\pi}{2}\,\mathrm{d}x=\pi\int_0^R(R^2-x^2)\,\mathrm{d}x=\pi\left[R^2x-\frac{1}{3}x^3\right]_0^R=\pi\left(R^2-\frac{1}{3}R^3\right)=\frac{2}{3}\pi R^3$$

Hence the volume of a sphere is twice this, $V_{\mathrm{sph}}=\frac{4}{3}\pi R^3$.

10) Calculate the volume of a sphere of radius R using a triple integral (in rectangular coordinates)

Solution:

Project the sphere onto the XZ axis by doing the integral along z: this produces a function $g(x,y)$ whose double integral can be done over the two dimensional disk.



$$\iiint_S f(x,y,z)\,\mathrm{d}V = \underbrace{\int_{-R}^{R}\left[\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \underbrace{\left(\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \mathrm{d}z\right)}_{g(x,y)} \mathrm{d}y\right]}_{h(z)} \mathrm{d}x$$

$$= \underbrace{\int_{-R}^{R}\left[\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \underbrace{\left(2\sqrt{R^2-x^2-y^2}\right)}_{g(x,y)} \mathrm{d}y\right]}_{h(z)} \mathrm{d}x$$

Which becomes identical to twice the double integral we did for the hemisphere. So, we can arrive once again at the answer $\frac{4}{3}\pi R^3$.

PROBLEMS

PROBLEMS: DOUBLE INTEGRATION IN NON-RECTANGULAR REGIONS

11) Calculate the integral $\iint_R xy\ \mathrm{d}A$ over the region R enclosed by the lines $x = 0$, $y = 0$ and $y = 1 - x$:

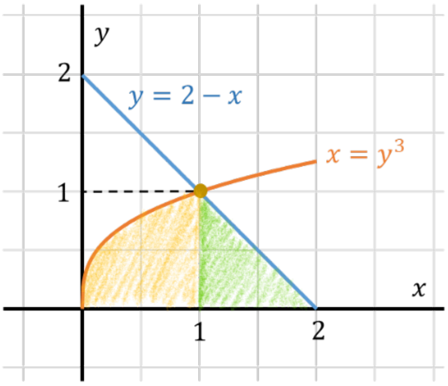
Solution: Integrating first along y:

$$\iint_R xy\ \mathrm{d}A = \int_0^1 \int_0^{1-x} xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left(\frac{1}{2}xy^2\right)_{y=0}^{y=1-x} \mathrm{d}x = \int_0^1 \left(\frac{1}{2}x(1-x)^2\right)\mathrm{d}x =$$

$$= \int_0^1 \left(\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right)\mathrm{d}x = \left[\frac{1}{8}x^4 - \frac{1}{3}x^3 + \frac{1}{4}x^2\right]_0^1 = \frac{1}{8} - \frac{1}{3} + \frac{1}{4} = \frac{1}{24}$$

12) Find the area of the region contained between the graphs $x = y^3$, $x + y = 2$ and $y = 0$.

Solution: First get intuition for the different graphs: $y = 0$ is the x-axis; $x = y^3$ is the inverse of $y = x^3$ which requires mirror-inverting it through the diagonal $x = y$; $x + y = 2$ is a straight-line crossing $(0,2)$ and $(2,0)$. The curve and the line intersect at $(1,1)$ (solution to their simultaneous equation $y^3 + y = 2$). Therefore, we draw the region:



We can do this integral in two different orders. If we decide to integrate along y first, the integral will need to be split into two, as the upper limit changes function half-way.

$$A_1 = \int_0^1 \left(\int_0^{\sqrt[3]{x}} 1\,\mathrm{d}y\right)\mathrm{d}x + \int_1^2 \left(\int_0^{2-x} 1\,\mathrm{d}y\right)\mathrm{d}x$$

$$A_2 = \int_0^1 \left(\int_{y^3}^{2-y} 1\,\mathrm{d}x\right)\mathrm{d}y$$

Both must give the same result:

$$A_1 = \int_0^1 x^{1/3}\,\mathrm{d}x + \int_1^2 (2-x)\mathrm{d}x = \left(\frac{3}{4}x^{\frac{4}{3}}\right)_{x=0}^{x=1} + \left(2x - \frac{x^2}{2}\right)_{x=1}^{x=2} = \frac{3}{4} + \left(4 - 2 - 2 + \frac{1}{2}\right) = \frac{5}{4}$$

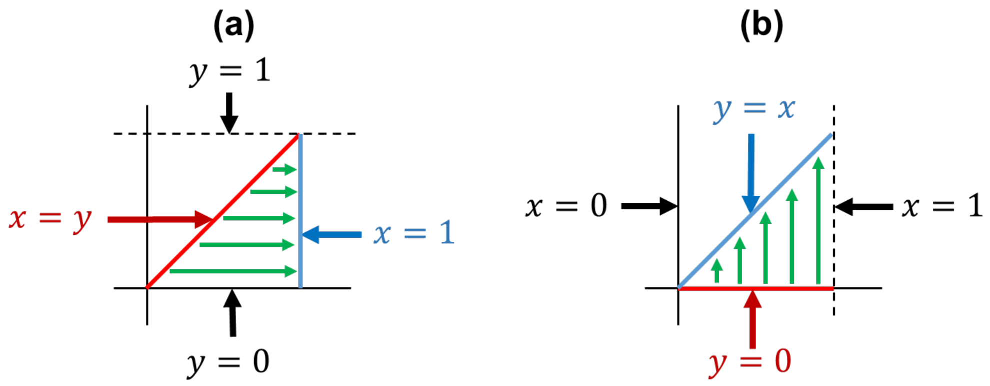
$$A_2 = \int_0^1 (x)_{x=y^3}^{x=2-y}\mathrm{d}y = \int_0^1 (2 - y - y^3)\mathrm{d}y = \left(2y - \frac{y^2}{2} - \frac{y^4}{4}\right)_{y=0}^{y=1} = \frac{5}{4}$$

13) Calculate

$$\iint_R \frac{\sin x}{x}\, \mathrm{d}A$$

Where R is the triangle in the XY plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution: Start by sketching the region of integration and identifying the limits.



(a) Integration along x first:

$$\iint_R \frac{\sin x}{x}\, \mathrm{d}A = \int_0^1 \left(\int_y^1 \frac{\sin x}{x}\, \mathrm{d}x \right) \mathrm{d}y$$

It cannot be done. There is no antiderivative for the function $\sin x\,/x$.

(b) Integration along y first:

$$\iint_R \frac{\sin x}{x}\, \mathrm{d}A = \int_0^1 \left(\int_0^x \frac{\sin x}{x}\, \mathrm{d}y \right) \mathrm{d}x$$

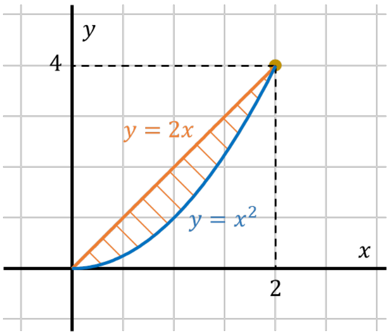
The inner integration is trivial because the function does not depend on y. Then, substituting the limits of integration helps to simplify the outer integral:

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left[\frac{\sin x}{x} y \right]_{y=0}^{y=x} \mathrm{d}x = \int_0^1 \left(\frac{\sin x}{x} x - 0 \right) \mathrm{d}x = \int_0^1 \sin x\ \mathrm{d}x = [-\cos x]_0^1$$

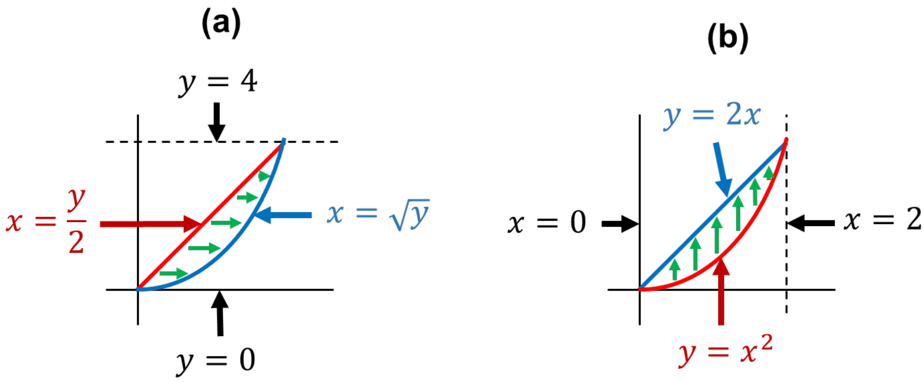
$$= 1 - \cos(1) \approx 0.46$$

14) Find $\iint_R (x^3 + 4y)\,\mathrm{d}A$ where R is the region of the plane contained between the graphs $y = x^2$ and $y = 2x$.

Solution: First, sketch the region and label the curves. The two graphs meet at $x = 0$ and $x = 2$ (the two simultaneous solutions).



We can choose two different orders of integration:



(a) Integrating first along x:

$$\iint_R (x^3 + 4y)\,\mathrm{d}A = \int_0^4 \left(\int_{y/2}^{\sqrt{y}} (x^3 + 4y)\mathrm{d}x \right) \mathrm{d}y = \int_0^4 \left(\frac{1}{4}x^4 + 4yx \right)_{x=y/2}^{x=\sqrt{y}} \mathrm{d}y$$

$$= \int_0^4 \left(\frac{y^2}{4} + 4y^{3/2} - \frac{y^4}{64} - 2y^2 \right) \mathrm{d}y = \int_0^4 \left(-\frac{7y^2}{4} - \frac{y^4}{64} + 4y^{3/2} \right) \mathrm{d}y$$

$$= \left(-\frac{7y^3}{4 \cdot 3} - \frac{y^5}{5 \cdot 64} + \frac{8y^{5/2}}{5} \right)_{y=0}^{y=4} = -\frac{112}{3} - \frac{16}{5} + \frac{256}{5} = \frac{32}{3}$$

(b) Integrating first along y:

$$\iint_R (x^3 + 4y)\,\mathrm{d}A = \int_0^2 \left(\int_{x^2}^{2x} (x^3 + 4y)\mathrm{d}y \right) \mathrm{d}x = \int_0^2 (x^3 y + 2y^2)_{y=x^2}^{y=2x} \mathrm{d}x$$

$$= \int_0^2 (\cancel{2x^4} + 8x^2 - x^5 - \cancel{2x^4})\mathrm{d}x = \left(\frac{8}{3}x^3 - \frac{1}{6}x^6 \right)_{x=0}^{x=2} = \frac{64}{3} - \frac{64}{6} = \frac{32}{3}$$

Both give the same result.

PROBLEMS: CHANGE OF VARIABLES

15) Let R be the region in the first quadrant of the xy-plane bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$ and $y = x + 1$. Evaluate the following double integral on R:

$$\iint_R (2x^2 - xy - y^2)\,\mathrm{d}x\,\mathrm{d}y$$

But do so by using the following change of variables to solve the integration:

$$\begin{cases} u = x - y \\ v = 2x + y \end{cases}$$

Solution: It is convenient to have the change of variables written in its two forms $(u, v) \to (x, y)$ and $(x, y) \to (u, v)$. Solving for x on the first equation and substituting on the second, and solving for y on the first and substituting on the second, we can arrive at the inverse change of variables:

$$\begin{cases} x = \dfrac{u + v}{3} \\ y = \dfrac{v - 2u}{3} \end{cases}$$

Now, to do a change of variables on a double integral, remember we need to do three things:

(i) Do the change of variables on the integrand $f(x, y) \to f(x(u, v), y(u, v))$.

(ii) Rewrite the limits in terms of the new variables (hopefully, this simplifies the limits)

(iii) Change the differential area into the new variables: $\mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}u\,\mathrm{d}v$

In order to be completely clear with the steps, I built a table:

	Coordinates x and y	**Coordinates u and v**
(i) Integrand	$2x^2 - xy - y^2$ $= (2x + y)(x - y)$	uv
(ii) Limits	$y = x - 2$	$u = 2$
	$y = x + 1$	$u = -1$
	$y = -2x + 4$	$v = 4$
	$y = -2x + 7$	$v = 7$
(iii) Differential of area (Jacobian determinant)	$\mathrm{d}x\,\mathrm{d}y$	$\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \mathrm{d}u\,\mathrm{d}v = \begin{vmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{vmatrix} \mathrm{d}u\,\mathrm{d}v = \frac{1}{3}\mathrm{d}u\,\mathrm{d}v$

Therefore, performing the three changes, we arrive at:

$$I = \iint_R (2x^2 - xy - y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_4^7 \int_{-1}^2 (uv)\frac{1}{3}\mathrm{d}u\,\mathrm{d}v$$

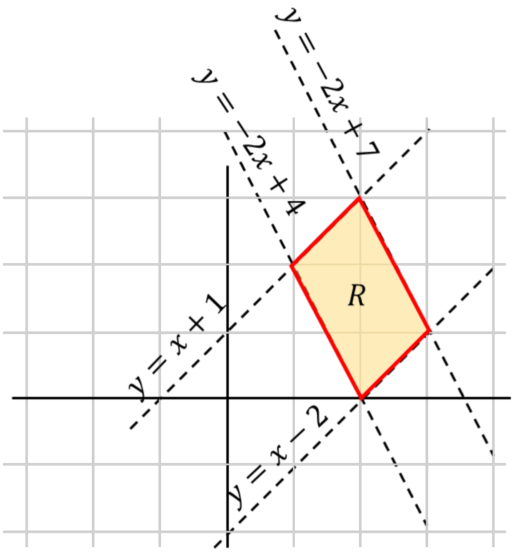
which is now an easy integral to be evaluated in a rectangular region. We can integrate first along u or first along v.

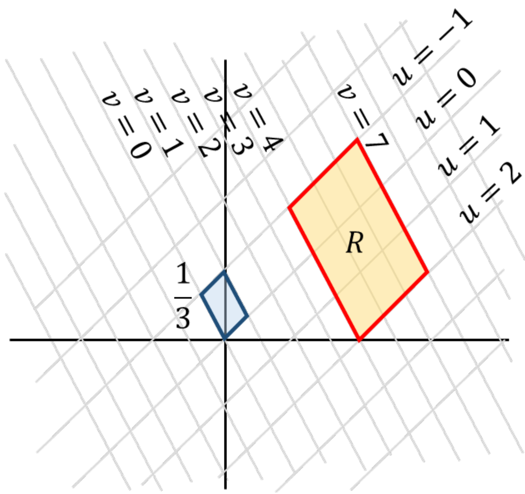
Integrating first along u:

$$I = \int_4^7 \int_{-1}^2 \left(\frac{uv}{3}\right) \mathrm{d}u\, \mathrm{d}v = \int_4^7 \left[\frac{u^2 v}{6}\right]_{u=-1}^{u=2} \mathrm{d}v = \int_4^7 \left(\frac{4v}{6} - \frac{v}{6}\right) \mathrm{d}v = \int_4^7 \frac{1}{2} v\, \mathrm{d}v = \left[\frac{1}{4} v^2\right]_4^7 = \frac{49}{4} - \frac{16}{4}$$
$$= \frac{33}{4}$$

Same result would be obtained integrating first along v.

We can visualize the region of integration and the new coordinate system. Notice why the change of variables allowed simplified limits. The system of coordinates $\{u, v\}$ is perfectly aligned with the region of integration, and hence the limits become very simple.





Also note how the Jacobian determinant of $1/3$ can be interpreted as the area of the parallelepiped formed by a unit increase in both u and v.

In this example, the change of coordinates was linear, so the new coordinates formed a regular grid. In general curvilinear coordinates, the new grid may be curved, and the Jacobian determinant may depend on the position, but the mathematical steps are identical.

16) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Evaluate the following double integral on the region R:

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) \mathrm{d}x\, \mathrm{d}y$$

Use the following change of variables to solve the integration:

$$\begin{cases} x = u/v \\ y = uv \end{cases}$$

with $u > 0$ and $v > 0$.

Solution: It is convenient to have the change of variables written in its two forms $(u, v) \to (x, y)$ and $(x, y) \to (u, v)$. Solving for u on the first equation and substituting on the second, and solving for v on the first and substituting on the second, we can arrive at the inverse change of variables:

$$\begin{cases} u = \sqrt{xy} \\ v = \sqrt{y/x} \end{cases}$$

Now, to do a change of variables on a double integral, remember we need to do three things:

(i) Do the change of variables on the integrand $f(x, y) \to f(x(u, v), y(u, v))$.

(ii) Rewrite the limits in terms of the new variables (hopefully, this simplifies the limits)

(iii) Change the differential area into the new variables: $\mathrm{d}x\, \mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\, \mathrm{d}u\, \mathrm{d}v$

Normally these can be done quickly. In order to be completely clear with the steps, I built a table:

	Coordinates x and y	**Coordinates u and v**
(i) Integrand	$\sqrt{\frac{y}{x}} + \sqrt{xy}$	$u + v$
(ii) Limits	$xy = 1$	$u = 1$
	$xy = 9$	$u = \sqrt{9} = 3$
	$y = x$	$v = 1$
	$y = 4x$	$v = \sqrt{4} = 2$
(iii) Differential of area (Jacobian determinant)	$\mathrm{d}x\, \mathrm{d}y$	$\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \mathrm{d}u\, \mathrm{d}v = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} \mathrm{d}u\, \mathrm{d}v = \frac{2u}{v} \mathrm{d}u\, \mathrm{d}v$

Therefore, performing the three changes, we arrive at:

$$I = \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) \mathrm{d}x\, \mathrm{d}y = \int_1^2 \int_1^3 (u + v) \frac{2u}{v} \mathrm{d}u\, \mathrm{d}v = \int_1^2 \int_1^3 \left(2\frac{u^2}{v} + 2u \right) \mathrm{d}u\, \mathrm{d}v$$

which is now an easy integral to be evaluated in a rectangular region. We can integrate first along u or first along v.

Integrating first along u:

$$I = \int_1^2 \int_1^3 \left(2\frac{u^2}{v} + 2u\right) \mathrm{d}u\, \mathrm{d}v = \int_1^2 \left[\frac{2}{3}\frac{u^3}{v} + u^2\right]_{u=1}^{u=3} \mathrm{d}v = \int_1^2 \left(\frac{18}{v} + 9 - \frac{2}{3v} - 1\right) \mathrm{d}v$$

$$= \int_1^2 \left(\frac{52}{3}v^{-1} + 8\right) \mathrm{d}v = \left[\frac{52}{3}\ln|v| + 8v\right]_{v=1}^{v=2} = \frac{52}{3}\ln 2 - \frac{52}{3}\ln 1 + 16 - 8$$

$$= \frac{52}{3}\ln 2 + 8$$

Alternatively, integrating first along v:

$$I = \int_1^3 \int_1^2 \left(2\frac{u^2}{v} + 2u\right) \mathrm{d}v\, \mathrm{d}u = \int_1^3 [2u^2 \ln|v| + 2uv]_{v=1}^{v=2}\, \mathrm{d}u$$

$$= \int_1^3 (2u^2 \ln 2 + 4u - 2u^2 \ln 1 - 2u)\, \mathrm{d}u = \int_1^3 (2u^2 \ln 2 + 2u)\, \mathrm{d}u$$

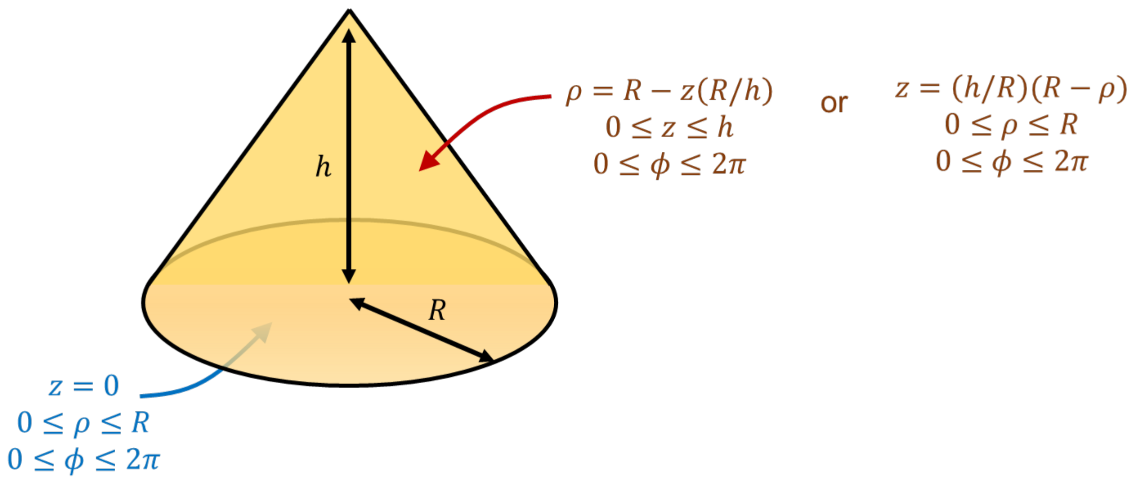
$$= \left[\frac{2}{3}u^3 \ln 2 + u^2\right]_{u=1}^{u=3} = \left(18\ln 2 + 9 - \frac{2}{3}\ln 2 - 1\right) = \frac{52}{3}\ln 2 + 8$$

PROBLEMS: MULTIPLE INTEGRATION IN POLAR/CYLINDRICAL/SPHERICAL COORDINATES

17) Calculate the volume of a cone of radius R and height h using triple integration.

Solution:

A cone is clearly well suited to integration in cylindrical coordinates. The first step is to figure out parametric equations for the surfaces of the cone:



These surfaces will be the limits of integration. We can choose different orders of integration. A simple one is to do first the integration on z, which must go from 0 to the equation for the curved surface. Do not forget the Jacobian for cylindrical coordinates when substituting the volume element $dV = \rho\, \mathrm{d}z\, \mathrm{d}\rho\, \mathrm{d}\phi$:

$$
\begin{aligned}
V = \iiint_R \mathrm{d}V &= \int_0^{2\pi}\int_0^R\int_0^{\frac{h}{R}(R-\rho)} \rho\, \mathrm{d}z\, \mathrm{d}\rho\, \mathrm{d}\phi = \int_0^{2\pi}\int_0^R [\rho z]_{z=0}^{z=\frac{h}{R}(R-\rho)}\, \mathrm{d}\rho\, \mathrm{d}\phi \\
&= \int_0^{2\pi}\int_0^R \left(h\rho - \frac{h}{R}\rho^2\right) \mathrm{d}\rho\, \mathrm{d}\phi = \int_0^{2\pi} \mathrm{d}\phi \int_0^R \left(h\rho - \frac{h}{R}\rho^2\right) \mathrm{d}\rho \\
&= (2\pi)\left[\frac{h}{2}\rho^2 - \frac{h}{3R}\rho^3\right]_0^R = (2\pi)\left(\frac{hR^2}{2} - \frac{hR^3}{3R}\right) = (2\pi)\frac{hR^2}{6} = \frac{h\pi R^2}{3}
\end{aligned}
$$

18) Calculate the electrostatic potential at $(x, y, z) = (0,0, h)$ for a uniformly charged disk of radius R with charge density σ. The disk lies in the XY plane and is centred at the origin. (Remember that the electrostatic potential Φ for a charge Q at a distance d is given by $\Phi = kQ/d$).

Solution:

First let's figure out what it is we are integrating. We want to obtain the potential, therefore we will integrate a differential element of potential:

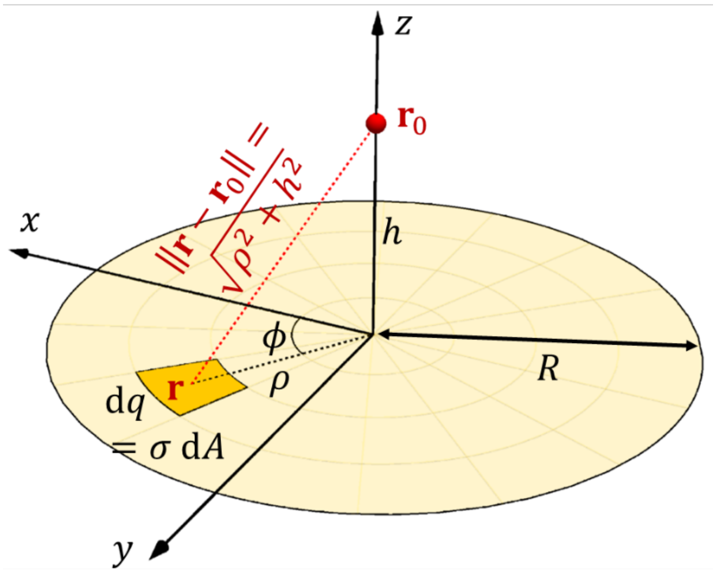
$$\Phi = \iint_S \mathrm{d}\Phi$$

Now, the differential element of potential corresponds to the potential created by a differential area $\mathrm{d}A$ of the disk. Such area will contain a differential charge $\mathrm{d}q = \sigma\,\mathrm{d}A$. Therefore:

$$\Phi(\mathbf{r}_0) = \iint_S \mathrm{d}\Phi = \iint_S k\frac{\mathrm{d}q}{\|\mathbf{r}-\mathbf{r}_0\|} = \iint_S k\frac{\sigma}{\|\mathbf{r}-\mathbf{r}_0\|}\mathrm{d}A$$

where $\|\mathbf{r}-\mathbf{r}_0\|$ is the distance between each location $\mathbf{r}$ on the surface and the observation point $\mathbf{r}_0$ where we are calculating the potential. From the point of view of the integral, $\mathbf{r} = (x, y, 0)$ is the position being integrated along the surface, while $\mathbf{r}_0$ is a constant. First, the symmetry of the problem strongly suggests that we should perform this integral in cylindrical coordinates, so that the disk is given by $\rho \in [0, R]$ and $\phi \in [0,2\pi]$. In cylindrical coordinates, the differential area is $\mathrm{d}A = \rho\,\mathrm{d}\phi\,\mathrm{d}\rho$.

A remaining issue is how to write $\|\mathbf{r}-\mathbf{r}_0\|$, the distance between the element $\mathrm{d}A$ and the observation point. This is clearly a function of the position within the disk, in fact we can write: $\mathbf{r}-\mathbf{r}_0 = (x, y, 0) - (0,0, h) = (x, y, -h)$. Therefore, $\|\mathbf{r}-\mathbf{r}_0\| = \sqrt{x^2+y^2+h^2} = \sqrt{\rho^2+h^2}$ which must be written in terms of cylindrical coordinates, as ρ and ϕ are our variables of integration.



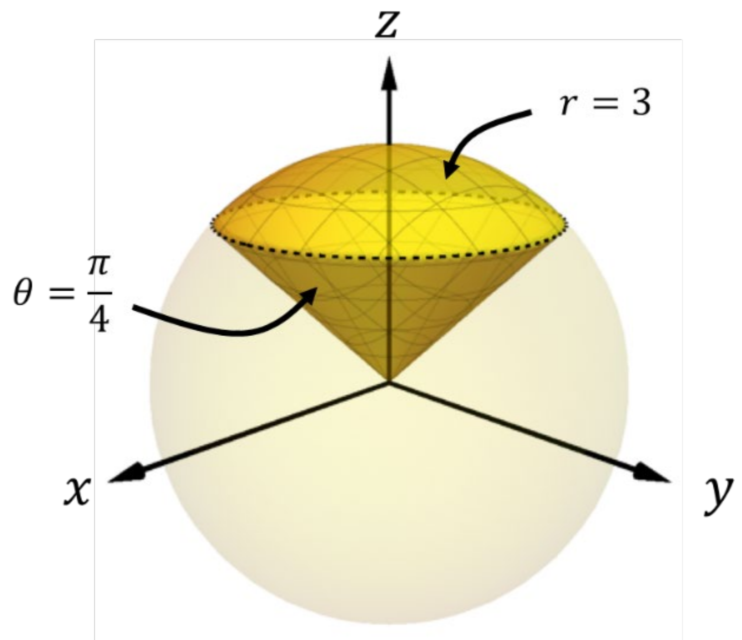
So, putting all together:

$$\Phi = \underbrace{\iint_S k\frac{1}{\sqrt{\rho^2+h^2}}\overbrace{\sigma\,\underbrace{\rho\,\mathrm{d}\phi\,\mathrm{d}\rho}_{\mathrm{d}A}}^{\mathrm{d}q}}_{\mathrm{d}\Phi} = k\sigma\left(\int_0^{2\pi}\mathrm{d}\phi\right)\left(\int_0^R \frac{\rho}{\sqrt{\rho^2+h^2}}\mathrm{d}\rho\right) = 2\pi k\sigma\left[(\rho^2+h^2)^{\frac{1}{2}}\right]_{\rho=0}^{\rho=R}$$

$$= 2\pi k\sigma\left(\sqrt{R^2+h^2}-h\right)$$

19) Calculate the volume enclosed by a sphere of radius $R \leq 3$ on the top, and the cone $z^2 \geq x^2 + y^2$ on the bottom.

Solution: Plot the region:



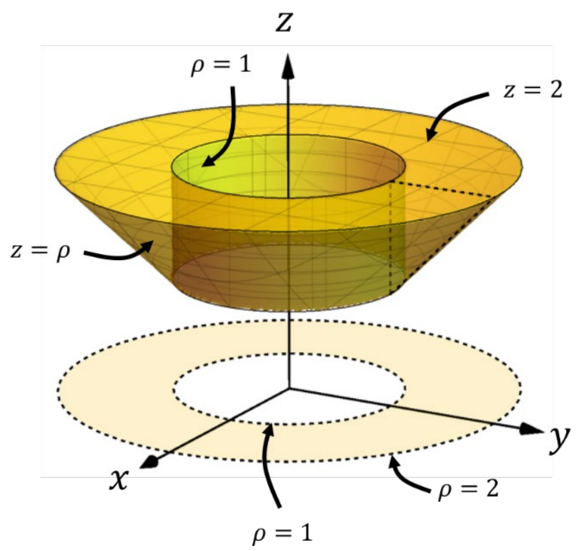
This integral is clearly adequate for spherical coordinates. Remember $\mathrm{d}V = r^2 \sin\theta \; \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$

$$V = \iiint_S 1 \, \mathrm{d}V = \iiint_S (r^2 \sin\theta) \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}\phi = \int_0^3 \left[\int_0^{2\pi} \left(\int_0^{\pi/4} \mathrm{r}^2 \sin\theta \, \mathrm{d}\theta \right) \mathrm{d}\phi \right] dr$$

$$= \left(\int_0^{2\pi} \mathrm{d}\phi \right) \left(\int_0^3 r^2 \, \mathrm{d}r \right) \left(\int_0^{\pi/4} \sin\theta \; \mathrm{d}\theta \right) = (2\pi)(9)[-\cos\theta]_{\theta=0}^{\theta=\pi/4} \; = 18\pi \left(1 - \frac{1}{\sqrt{2}} \right)$$

20) Calculate the volume of the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$, the planes $z = 1$ and $z = 2$, and outside the cylinder $x^2 + y^2 = 1$.

Solution: Let's plot the region:



The integration in rectangular coordinates would be very messy. In **cylindrical coordinates** it's easy. Don't forget including the Jacobian determinant ρ in the integrand.

$$V = \iiint_R 1 \, \mathrm{d}V = \iiint_R \rho \, \mathrm{d}z \, \mathrm{d}\rho \, \mathrm{d}\phi = \int_0^{2\pi} \left(\int_1^2 \left(\int_\rho^2 \rho \, \mathrm{d}z \right) \mathrm{d}\rho \right) \mathrm{d}\phi = \left(\int_0^{2\pi} \mathrm{d}\phi \right) \int_1^2 (\rho z)_{z=\rho}^{z=2} \, \mathrm{d}\rho$$

$$= 2\pi \int_1^2 (2\rho - \rho^2) \, \mathrm{d}\rho = 2\pi \left(\rho^2 - \frac{1}{3}\rho^3 \right)_{\rho=1}^{\rho=2} = 2\pi \left(4 - \frac{8}{3} - 1 + \frac{1}{3} \right) = \frac{4}{3}\pi$$

4.3 SURFACE INTEGRALS

The double integrals we have considered so far were always limited to **PLANAR** regions. What if we want to integrate over a curved surface? For example: I want to calculate the average temperature of the surface of the sun. Or even simpler, I want to calculate the surface area of some surface.

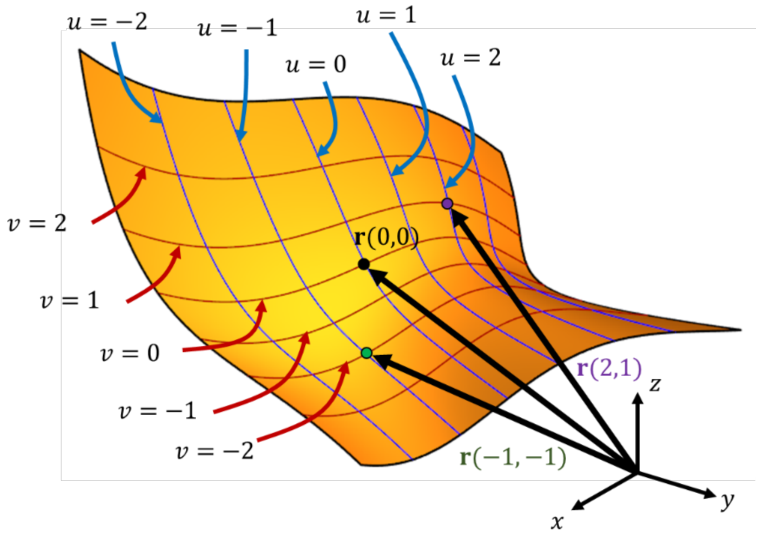
A. PARAMETRIZED SURFACES

A surface in 3D space can be specified via a function

$\mathbf{r}: (u, v) \mapsto (x, y, z)$

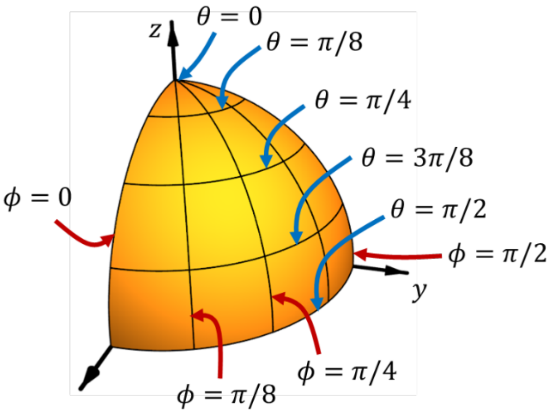
which maps two parameters u and v to points in 3-D space $\mathbf{r}(u, v)$.

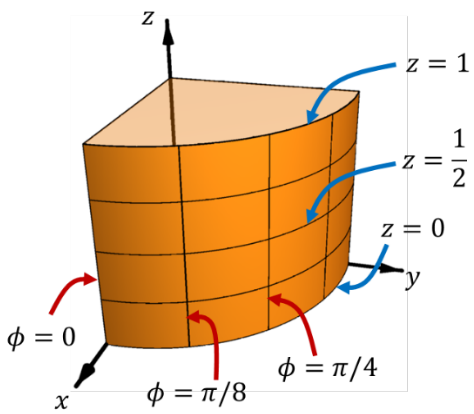
The parameters u and v represent the two degrees of freedom of a surface.



To be formal, we must require that each unique value of (u, v) results in a single position (x, y, z). On the contrary, we can allow a same (x, y, z) to be represented by several values of (u, v) (i.e. surfaces may cut across themselves).

Examples of surface parametrizations:





$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{pmatrix}$ **spherical surface**

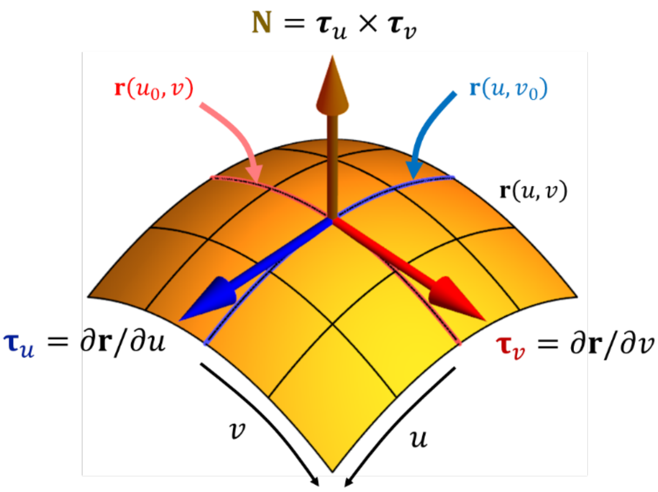
$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ z \end{pmatrix}$ **cylindrical surface**

CURVES IN THE SURFACE, TANGENT VECTORS, NORMAL VECTOR

If we fix $u = u_0$, the resulting function $\mathbf{r}(u_0, v): v \mapsto (x, y, z)$ defines a curve in the surface. Similarly, $\mathbf{r}(u, v_0)$ defines another curve in the surface.

The tangent vectors to these curves at the location $\mathbf{r}(u, v)$ are therefore given by:

$$\boldsymbol{\tau}_u(u, v) = \partial\mathbf{r}/\partial u$$
$$\boldsymbol{\tau}_v(u, v) = \partial\mathbf{r}/\partial v$$



These are two vectors always tangent to the surface, at every location in the surface (u, v). Also note that these two vectors form the two columns of the Jacobian matrix for $\mathbf{r}: (u, v) \mapsto (x, y, z)$.

We say the surface is smooth if the functions $\mathbf{r}(u, v)$, $\boldsymbol{\tau}_u(u, v)$, and $\boldsymbol{\tau}_v(u, v)$ are continuous, and the latter two are non-parallel everywhere inside the domain defined for u and v. The normal vector to the surface must be given by the cross product of the two tangent vectors:

$$\mathbf{N} = \boldsymbol{\tau}_u \times \boldsymbol{\tau}_v$$

We can express this by components in terms of Jacobian determinants:

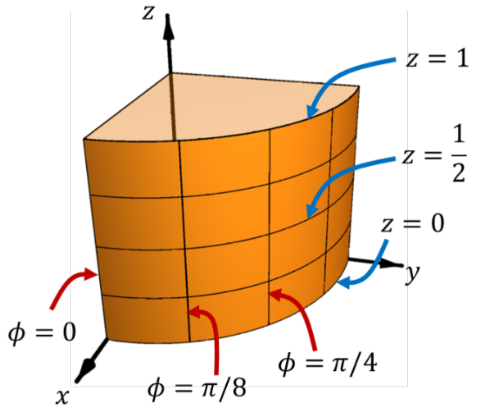
$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\mathbf{N} = \hat{\mathbf{x}} \underbrace{\frac{\partial(y, z)}{\partial(u, v)}}_{J_x(u,v)} + \hat{\mathbf{y}} \underbrace{\frac{\partial(z, x)}{\partial(u, v)}}_{J_y(u,v)} + \hat{\mathbf{z}} \underbrace{\frac{\partial(x, y)}{\partial(u, v)}}_{J_z(u,v)}$$

So, **the normal is expressed via the Jacobians, which form its components**. Note that the arguments in the nominators of the Jacobians together with the unit vectors form the cyclic sequence $x \to y \to z \to x \to y \to \cdots$. This helps to memorise this formula.

1) Calculate the normal vector $\mathbf{N}$ to the outside surface of a cylinder.

The surface can be parametrized via $\mathbf{r}(\phi, z)$ where $\mathbf{r}$ is given by its components $x = R\cos\phi$; $y = R\sin\phi$; $z = z$ with parameters $\phi \in [0,2\pi]$ and $z \in [0, h]$.



$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ z \end{pmatrix}$$

Let's consider the curves defined by keeping each of the two parameters constant:

$\mathbf{r}(\phi_0, z) = \mathbf{r}_{\phi_0}(z) = \begin{pmatrix} R\cos\phi_0 \\ R\sin\phi_0 \\ z \end{pmatrix}$ defines a vertical line parallel to z in the surface.

The tangential vector to this curve is $\rightarrow \boldsymbol{\tau}_z(\phi) = \frac{\partial\mathbf{r}(\phi_0,z)}{\partial z} = (0,0,1) = \hat{\mathbf{z}}$

$\mathbf{r}(\phi, z_0) = \mathbf{r}_{z_0}(\phi) = \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ z_0 \end{pmatrix}$ defines a circle around the cylinder at height z_0.

The tangential vector to this curve is $\rightarrow \boldsymbol{\tau}_\phi(z) = \frac{\partial\mathbf{r}(\phi,z_0)}{\partial\phi} = (-R\sin\phi\,, R\cos\phi\,, 0) = R\hat{\mathbf{e}}_\phi$

Therefore, the normal vector $\mathbf{N} = \boldsymbol{\tau}_\phi \times \boldsymbol{\tau}_z = R\hat{\mathbf{e}}_\phi \times \hat{\mathbf{z}} = R\hat{\mathbf{e}}_\rho$ as one would expect.

Alternatively, we could have simply applied the direct recipe for $\mathbf{N}$:

$$\mathbf{N} = \hat{\mathbf{x}}\begin{vmatrix} \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \\ \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} + \hat{\mathbf{y}}\begin{vmatrix} \frac{\partial z}{\partial\phi} & \frac{\partial x}{\partial\phi} \\ \frac{\partial z}{\partial z} & \frac{\partial x}{\partial z} \end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix} \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \end{vmatrix} = \hat{\mathbf{x}}\begin{vmatrix} R\cos\phi & 0 \\ 0 & 1 \end{vmatrix} + \hat{\mathbf{y}}\begin{vmatrix} 0 & -R\sin\phi \\ 1 & 0 \end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix} \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} \\ 0 & 0 \end{vmatrix}$$
$$= R\cos\phi\,\hat{\mathbf{x}} + R\sin\phi\,\hat{\mathbf{y}} = R\hat{\mathbf{e}}_\rho$$

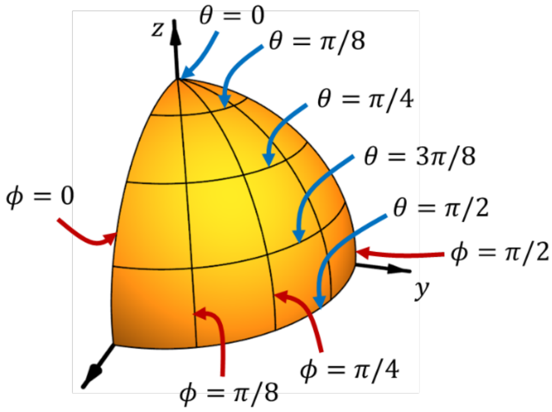
Note that we could say that any multiple of this vector is normal to the surface, so for example we could normalize it and say that the unit normal vector is $\hat{\mathbf{n}} = \hat{\mathbf{e}}_\rho$.

However, the specific vector $\mathbf{N}$ with its specific scaling factor R will be important later. So we will reserve the capital letter notation $\mathbf{N}$ for this special vector without normalization.

2) Calculate the normal vector $\mathbf{N}$ to the outside surface of a sphere.

The spherical surface can be parametrized via $\mathbf{r}(\phi,\theta)$ where $\mathbf{r}$ is given by:

$\mathbf{r}(\phi,\theta) = \begin{pmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{pmatrix}$ with parameters $\phi \in [0,2\pi]$ and $\theta \in [0,\pi]$.



Let's consider the curves defined by keeping each of the two parameters constant.

If we keep $\phi = \phi_0$ constant and vary θ we arrive at $\mathbf{r}(\phi_0,\theta)$, which define the great circles which cross both poles $z = \pm R$.

The tangential vector to these curves is $\rightarrow \boldsymbol{\tau}_\phi(\theta) = \frac{\partial \mathbf{r}(\phi,\theta)}{\partial\theta} = \begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} = R\hat{\mathbf{e}}_\theta$

If we keep $\theta = \theta_0$ constant and vary ϕ we arrive at $\mathbf{r}(\phi,\theta_0)$, which define circles parallel to the equator and becoming smaller as you approach the poles $z = \pm R$.

The tangential vector to these curves is:

$$\rightarrow \boldsymbol{\tau}_\theta(\phi) = \frac{\partial \mathbf{r}(\phi,\theta)}{\partial\phi} = \begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix} = R\sin\theta \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} = R\sin\theta\,\hat{\mathbf{e}}_\phi$$

Therefore, the normal vector $\mathbf{N} = \boldsymbol{\tau}_\phi \times \boldsymbol{\tau}_\theta = R\hat{\mathbf{e}}_\theta \times R\sin\theta\,\hat{\mathbf{e}}_\phi = R^2\sin\theta\,\hat{\mathbf{e}}_r$, in the direction of the radial unit vector, as one would have expected.

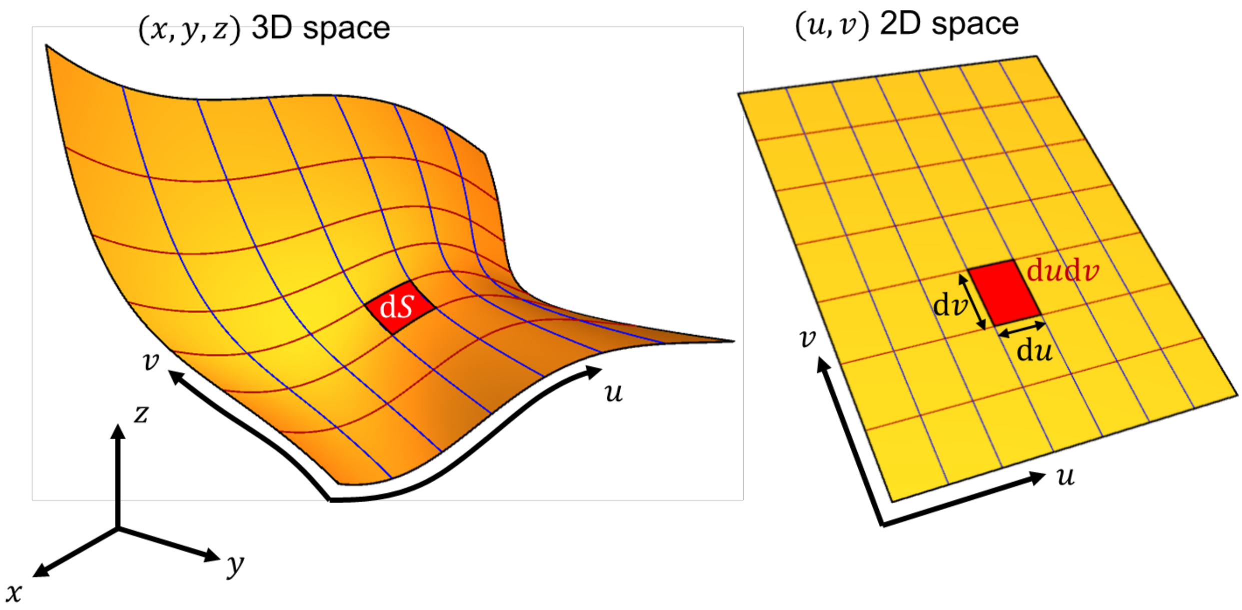
Alternatively, we could have followed the recipe:

$$\mathbf{N} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial\phi} & \frac{\partial x}{\partial\phi} \\ \frac{\partial z}{\partial\theta} & \frac{\partial x}{\partial\theta} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} \end{vmatrix} = (\dots) = R^2\sin\theta\,\hat{\mathbf{e}}_r$$

B. SURFACE INTEGRAL (DOUBLE INTEGRAL OVER A SURFACE)

When we have done a double integral in the past, we always integrated over a planar region, so how are we supposed to do an integration over a surface or region that is curved?

The answer is that we don't need to! We can map the curved surface into a planar region, by simply using the plane of the parameters (u, v), and we can carry on with the double integral as usual, as long as the surface area element $\mathrm{d}S$ is properly chosen to match the actual surface element on which the little square $\mathrm{d}u\,\mathrm{d}v$ in parameter space (u, v) maps to in the actual surface in (x, y, z) space.



This should remind you greatly of the change of variables.

The surface element can in general depend on the position $\mathrm{d}S = g(u, v)\,\mathrm{d}u\,\mathrm{d}v$.

So, we can perform the surface integral as a double integral in the space of the parameters u and v.

$$\iint_S f(u, v)\,\mathrm{d}S = \int_{v_1}^{v_2} \int_{u_1(v)}^{u_2(v)} f(u, v)\, \underbrace{g(u, v)\mathrm{d}u\,\mathrm{d}v}_{\mathrm{d}S}$$

As a special case, the area of the surface S can be calculated by integrating with unit integrand:

$$A = \iint_S 1\,\mathrm{d}S$$

But how do we **find the surface element** $\mathrm{d}S$? Like in change of variables, we have both an intuitive way and a formal recipe to find the surface element $\mathrm{d}S$.

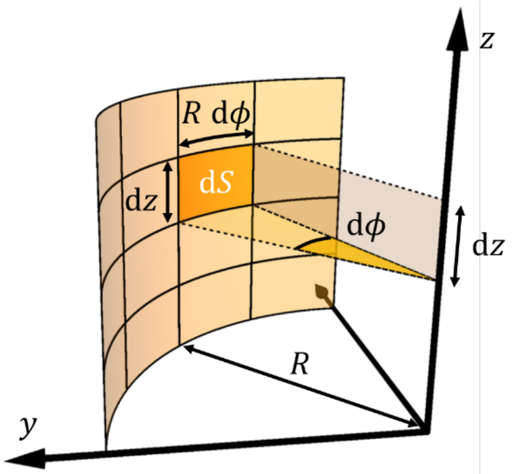
OPTION 1: FIND $\mathrm{d}S$ BY GEOMETRICAL INTUITION

Useful most of the time for simple surfaces: cylinders, spheres, cones.

Draw, or imagine, being at any point in your surface defined by the two parameters (u, v). Then calculate the area $\mathrm{d}S$ which is swept when $u \to u + du$ and $v \to v + dv$. In the limit of small differentials, the area swept becomes a parallelogram.

3) Calculate the area of the **outer curved surface of a cylinder** of radius R and height H by doing a surface integral

Solution: Our two parameters can be the angle and the height. Now imagine, or draw, the little "curved rectangle" created by an increase in angle $d\phi$ and an increase in height dz. Remember that we define the angle ϕ as subtended from the z-axis, not the origin



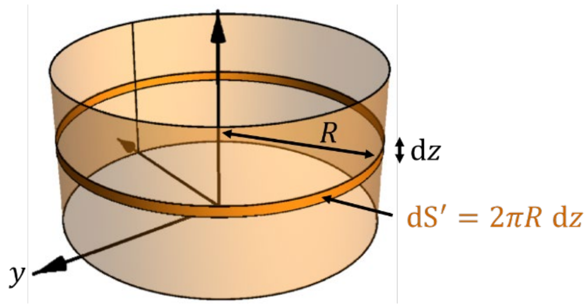
The little rectangle will, in the limit of small differentials, become a planar surface. Clearly the area is the product of its sides, so $\mathrm{d}S = R\,\mathrm{d}\phi\,\mathrm{d}z$.

Therefore, we can perform the integration as a simple double integration in the two dimensional space of variables (ϕ, z):

$$A = \iint_S \mathrm{d}S = \iint_{S(\phi z)} \overbrace{R\,\mathrm{d}\phi\,\mathrm{d}z}^{\mathrm{d}S} = \int_0^H \left(\int_0^{2\pi} R\,\mathrm{d}\phi\right) \mathrm{d}z = R\int_0^H \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi = 2\pi R H$$

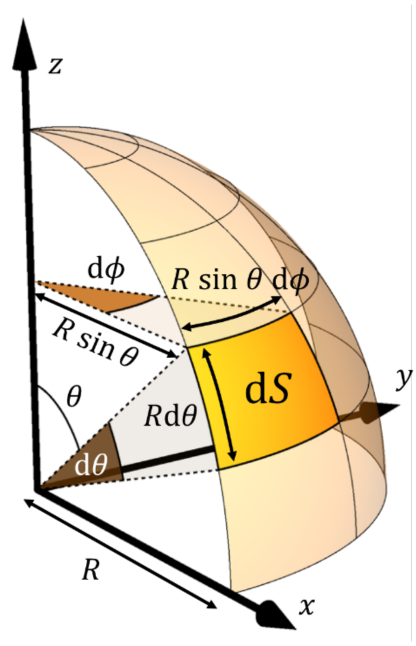
Alternative method: notice that if we carry out the integral in ϕ first, we can interpret the result as a single integration of a new circular surface element $\mathrm{d}S'$ (shown below). But this would not have been possible if some of the limits depended on ϕ, or if we had used an integrand that depended on ϕ.

$$\int_0^H \left(\int_0^{2\pi} R\,\mathrm{d}\phi\right) \mathrm{d}z = \int_0^H 2\pi R\,\mathrm{d}z = \int_0^H \mathrm{d}S'$$



4) Calculate the surface area of a sphere of radius R.

Solution: Our two parameters can be the azimuthal angle ϕ and the elevation angle θ commonly used in spherical coordinates. Now imagine, or draw, the little "curved rectangle" created by an increase in angle $d\phi$ and a simultaneous increase in angle $d\theta$. Remember that **the angle** $d\theta$ is **subtended at the origin**, **while the angle** $d\phi$ **is subtended at the z-axis** (by definition of the spherical coordinates)



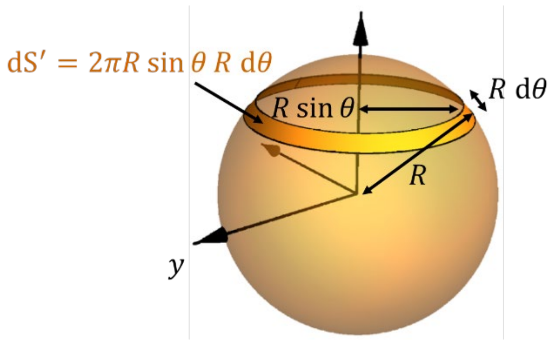
The little rectangle will, in the limit of small differentials, become a planar surface. The area of the rectangle will be the product of its sides, so $\mathrm{d}S = (R\sin\theta\ d\phi)(R\ d\theta) = R^2 \sin\theta\ \mathrm{d}\phi\ \mathrm{d}\theta$.

Therefore, we can perform the integration as a simple double integration in the plane of (ϕ, z):

$$A = \iint_S \mathrm{d}S = \iint_{S_{(\phi z)}} R^2 \sin\theta\, \mathrm{d}\phi\, \mathrm{d}\theta = R^2 \left(\int_0^{2\pi} \mathrm{d}\phi\right)\left(\int_0^{\pi} \sin\theta\, \mathrm{d}\theta\right) = 2\pi R^2 (-\cos\theta)_{\theta=0}^{\theta=\pi} = 4\pi R^2$$

Alternative interpretation: If we carry out the integral in ϕ first, the resulting area can be interpreted as a single integral of a circular surface element:

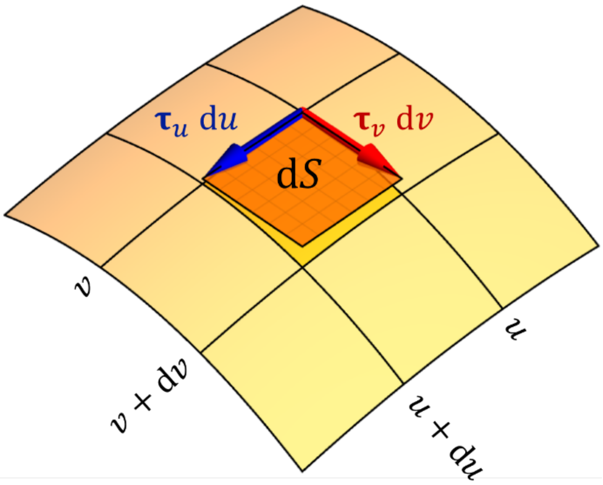
$$\int_0^{\pi}\int_0^{2\pi} R^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi = \int_0^{\pi} (2\pi R\sin\theta)(R\, \mathrm{d}\theta) = \int_0^{\pi} \mathrm{d}S'$$



That's it! The surface of a sphere was the problem which Archimedes was **most** proud to have solved!

OPTION 2: FIND $\mathrm{d}S$ WITH THE NORM OF NORMAL VECTOR $\mathbf{N}$ (GENERAL RECIPE)

In some cases, the curve can be too complicated to find $\mathrm{d}S$ by intuition. In this case we can follow a recipe, which always works.



The area of the surface formed by an increase $u \to u + du$ and $v \to v + dv$ is actually the area of the parallelogram formed by the two vectors $\boldsymbol{\tau}_u \mathrm{d}u$ and $\boldsymbol{\tau}_v \mathrm{d}v$, where $\boldsymbol{\tau}_u$ and $\boldsymbol{\tau}_v$ are the tangent vectors defined earlier using partial derivatives. As we well know, the area of this parallelogram is given by the absolute value of the cross product. Therefore $\mathrm{d}S = \|(\boldsymbol{\tau}_u \mathrm{d}u\,) \times (\boldsymbol{\tau}_v \mathrm{d}v)\|$, which can be written by taking the factors $\mathrm{d}u$ and $\mathrm{d}v$ outside:

$$\boxed{\mathrm{d}S = \|\boldsymbol{\tau}_u \times \boldsymbol{\tau}_v\|\,\mathrm{d}u\,\mathrm{d}v = \|\mathbf{N}\|\,\mathrm{d}u\,\mathrm{d}v}$$

Where $\mathbf{N}$ is the same normal vector defined earlier using the Jacobians. I have seen many books write this in an unnecessarily scary way, by explicitly writing everything directly, as follows: “A double integral of an integrand $f(u,v)$ over a surface parametrized by (u,v) is given by:

$$I = \iint_S f(u,v)\,\mathrm{d}S = \iint_{S_{(uv)}} f(u,v)\sqrt{\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2}\,\mathrm{d}u\,\mathrm{d}v$$

Notice that this form has already substituted $\|\mathbf{N}\| = \sqrt{N_x^2 + N_y^2 + N_z^2}$

Insight: Look at the equation above. The change of variables in double integrals in planar regions using the Jacobian in the previous lesson is nothing else than a special case of the current general equation, when $\mathbf{N}$ has only a $\hat{\mathbf{z}}$ component because the surface of integration is in the XY plane.

5) Find $\mathrm{d}S$ for the curved surface of a cylinder using the general recipe

Solution: From an earlier problem, after some tedious steps we found:

$$\mathbf{N} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial \phi} & \frac{\partial x}{\partial \phi} \\ \frac{\partial z}{\partial z} & \frac{\partial x}{\partial z} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} R\cos\phi & 0 \\ 0 & 1 \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} 0 & -R\sin\phi \\ 1 & 0 \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \\ 0 & 0 \end{vmatrix}$$
$$= R\cos\phi\,\hat{\mathbf{x}} + R\sin\phi\,\hat{\mathbf{y}} = R\hat{\mathbf{e}}_\rho$$

Therefore, the recipe gives $\mathrm{d}S = \|\mathbf{N}\|\,\mathrm{d}\phi\,\mathrm{d}z = R\,\mathrm{d}\phi\,\mathrm{d}z$, in agreement with our intuitive derivation.

6) Find $\mathrm{d}S$ for the surface of a sphere using the general recipe

Solution: From an earlier problem, after some tedious steps we found:

$$\mathbf{N} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial \phi} & \frac{\partial x}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial x}{\partial \theta} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = (\ldots) = R^2 \sin\theta\,\hat{\mathbf{e}}_r$$

Therefore, the recipe tells us $dS = \|\mathbf{N}\|\,\mathrm{d}\phi\,\mathrm{d}z = R^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}z$, in agreement with our intuitive derivation.

PARAMETRIZATION OF A SURFACE GIVEN AS A MATHEMATICAL FUNCTION $z = f(x, y)$

Often in maths, a surface is given as a function $z = f(x, y)$. How can we write that as a parametrized function $\mathbf{r}(u, v)$ in order to do the integration?

A possible answer is very simple: use x and y as the two parameters.

$$\mathbf{r}(u, v) = \begin{pmatrix} u \\ v \\ f(u, v) \end{pmatrix}$$

7) Calculate the surface area of the cone given by $z = \sqrt{x^2 + y^2}$ cut between the planes $z = 0$ and $z = R$.

Solution: First, picture what this cone looks like. It is a cone at 45 degrees with the z axis, i.e. with a slope of 1. Now, we need to parametrize this cone.

Option 1: Use x and y as parameters $\mathbf{r}(x, y) = \left(x, y, \sqrt{x^2 + y^2}\right)$. In that case, the normal vector $\mathbf{N}$ is:

$$\mathbf{N} = \hat{\mathbf{x}} \underbrace{\frac{\partial(y,z)}{\partial(x,y)}}_{J_x(x,y)} + \hat{\mathbf{y}} \underbrace{\frac{\partial(z,x)}{\partial(x,y)}}_{J_y(x,y)} + \hat{\mathbf{z}} \underbrace{\frac{\partial(x,y)}{\partial(x,y)}}_{J_z(x,y)} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial x}{\partial x} \\ \frac{\partial z}{\partial y} & \frac{\partial x}{\partial y} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} \end{vmatrix}$$

$$= \hat{\mathbf{x}} \begin{vmatrix} 0 & x(x^2+y^2)^{-1/2} \\ 1 & y(x^2+y^2)^{-1/2} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} x(x^2+y^2)^{-1/2} & 1 \\ y(x^2+y^2)^{-1/2} & 0 \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -\frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}}{\sqrt{x^2+y^2}} + \hat{\mathbf{z}}$$

Therefore, the surface element is given by:

$$\mathrm{d}S = \|\mathbf{N}\|\,\mathrm{d}x\,\mathrm{d}y = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y$$

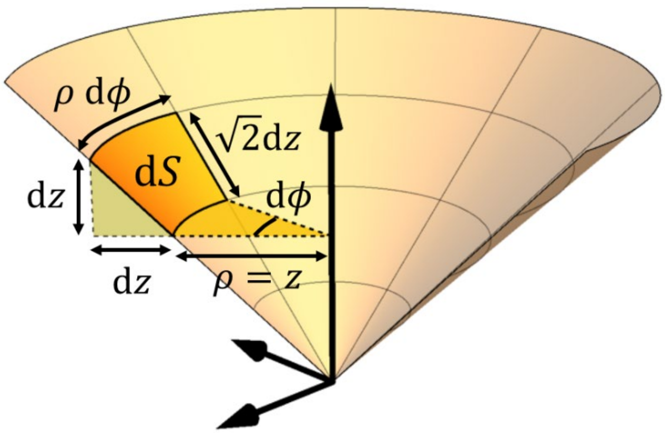
And so the integral is:

$$A = \iint_S \mathrm{d}S = \iint_{\mathrm{disk}} \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2} \iint_{\mathrm{disk}} \mathrm{d}x\,\mathrm{d}y = \sqrt{2}\pi R^2$$

Option 2: Figure out some other way to parametrize the surface. For example, we could use the height z as one parameter, and the polar angle ϕ as another, and then think how to parametrize the cone. This requires some visual intuition. For example:

$$\mathbf{r}(z, \phi) = \begin{pmatrix} z\cos\phi \\ z\sin\phi \\ z \end{pmatrix}$$

We can now figure out the surface element by doing $\mathrm{d}S = \|\mathbf{N}\|\,\mathrm{d}z\,\mathrm{d}\phi$, or alternatively, with a good imagination, you can figure it out intuitively: $\mathrm{d}S = (z\,\mathrm{d}\phi)(\sqrt{2}\,\mathrm{d}z)$. Can you explain why?



Therefore, the integral is:

$$A = \iint_S \mathrm{d}S = \iint_{S_{(z,\phi)}} (z\sqrt{2})\,\mathrm{d}z\,\mathrm{d}\phi = \sqrt{2}\left(\int_0^R z\,\mathrm{d}z\right)\left(\int_0^{2\pi} \mathrm{d}\phi\right) = \sqrt{2}\pi R^2$$

C. SURFACE INTEGRATION OF SCALAR AND VECTOR FIELDS

In the previous examples we were calculating areas of the surface, i.e. the integrand was $1\,\mathrm{d}S$. In general, we can also integrate some scalar field on the surface, i.e.

$$I = \iint_S f(\mathbf{r})\,\mathrm{d}S$$

Still, we need to obtain the appropriate $\mathrm{d}S$ for a parametrized surface and proceed as with a normal double integration.

We could define the following steps for surface integration (with practice, you should be able to stop thinking about steps and do them automatically):

$$\iint_S f(\mathbf{r})\,\mathrm{d}S$$

(1) Define **parametrization of the surface** S

$$\mathbf{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

(2) Obtain the **surface differential**

(i) Remember it, or look it up (for typical spherical and cylindrical surfaces)

(ii) Obtain it from geometrical intuition

(iii) Obtain it from the normal vector to the surface $\mathbf{N}$

$$\mathrm{d}S(u,v) = \|\mathbf{N}(u,v)\|\mathrm{d}u\,\mathrm{d}v = \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| \mathrm{d}u\,\mathrm{d}v$$

$$= \left\| \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right\| \mathrm{d}u\,\mathrm{d}v$$

(3) **Evaluate the integrand <u>at the surface</u>** (as a function of the parameters)

$$f(\mathbf{r}) \xrightarrow[\mathbf{r}(u,v)]{} f(u,v)$$

(4) **Perform the double integral** in the (u,v) plane [*use appropriate limits of planar double integral such that the parameters cover the whole surface S*]

$$\iint_S f(\mathbf{r})\,\mathrm{d}S = \int_{v_1}^{v_2} \int_{u_1(v)}^{u_2(v)} f(u,v)\, \underbrace{\|\mathbf{N}(u,v)\|\mathrm{d}u\,\mathrm{d}v}_{\mathrm{d}S}$$

8) Calculate the surface integral $\iint_S x\, \mathrm{d}S$ where the surface S is the part of the sphere $x^2 + y^2 + z^2 = R^2$ in which $x > 0$, $y > 0$ and $z > 0$ (i.e. the first octant).

Solution:

(1) Define **parametrization of the surface**

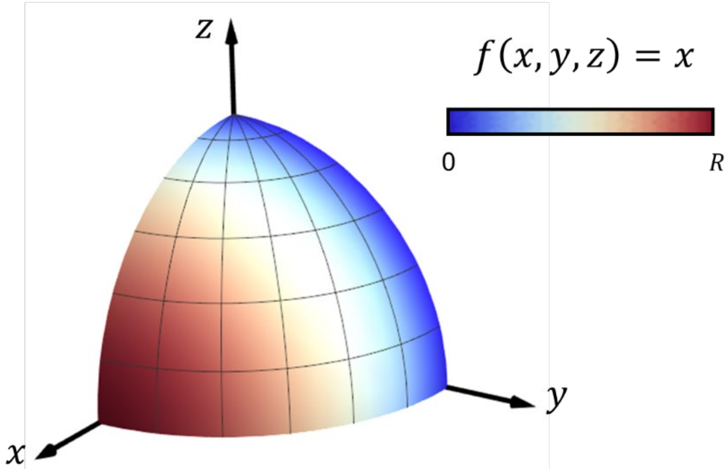
This surface is easy to parametrize if we use spherical coordinates:

$\mathbf{r}(\theta, \phi) = (R \sin\theta \cos\phi\,, R \sin\theta \sin\phi\,, R\cos\phi)$ with: $\phi \in [0, \pi/2]$ and $\theta \in [0, \pi/2]$ to cover only the first octant.

(2) Obtain the **surface differential**

As we derived in a previous exercise, the spherical surface element is given by $\mathrm{d}S = R^2 \sin\theta\ \mathrm{d}\phi\, \mathrm{d}\theta$.

(3) **Evaluate the integrand at the surface** (as a function of the parameters)



The function $f(\mathbf{r}) = x$ changes value throughout the surface, so to perform the integral **we cannot treat it as a constant**, we need to **evaluate the integrand ON the surface**, i.e. substitute the value of x from the surface parametrization $\mathbf{r} = (R\sin\theta\cos\phi\,, R\sin\theta\sin\phi\,, R\cos\phi)$, to write it in terms of the parameters:

$$f(\mathbf{r}) = x \quad \xrightarrow[\mathbf{r}(\theta,\phi)]{} \quad f(\theta, \phi) = R\sin\theta\cos\phi$$

(4) **We are ready to perform the required integral** in the (θ, ϕ) plane:

$$I = \iint_S x\, \mathrm{d}S = \iint_{S_{(\phi\theta)}} \underbrace{(R\sin\theta\cos\phi)}_{x} \underbrace{(R^2\sin\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta}_{\mathrm{d}S} = R^3 \left(\int_0^{\frac{\pi}{2}} \sin^2\theta\ \mathrm{d}\theta\right)\left(\int_0^{\frac{\pi}{2}} \cos\phi\ \mathrm{d}\phi\right)$$

$$= R^3 \left(\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\ \mathrm{d}\theta\right)\left(\int_0^{\frac{\pi}{2}} \cos\phi\ \mathrm{d}\phi\right) = R^3 \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} [\sin\phi]_{\phi=0}^{\phi=\frac{\pi}{2}}$$

$$= R^3\left(\frac{\pi}{4} - 0 - 0 + 0\right)(1 - 0) = \frac{\pi R^3}{4}$$

We can also integrate a **vector** over a surface, the result being another vector:

$$\mathbf{I} = \iint_S \mathbf{f}(u,v)\ \mathrm{d}S$$

We just do the integral separately for each component of the vector. This is simple to understand as follows (linearity of the integral and taking the basis vectors outside):

$$\begin{aligned}\mathbf{I} = \iint_S \mathbf{f}(u,v)\ \mathrm{d}S &= \iint_S (f_x(u,v)\hat{\mathbf{x}} + f_y(u,v)\hat{\mathbf{y}} + f_z(u,v)\hat{\mathbf{z}})\ \mathrm{d}S \\ &= \hat{\mathbf{x}} \iint_S f_x(u,v)\ \mathrm{d}S + \hat{\mathbf{y}} \iint_S f_x(u,v)\ \mathrm{d}S + \hat{\mathbf{z}} \iint_S f_x(u,v)\ \mathrm{d}S\end{aligned}$$

Just one warning: be careful when integrating vectors in <u>cylindrical or spherical BASIS</u>, because, as you know, the unit vectors in those bases depend on position, and therefore cannot be taken outside the integrals as constants. So, you cannot integrate the components in cylindrical or spherical coordinates separately. When integrating such vectors, it is convenient to convert them to **rectangular <u>basis</u>** first, so that $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}, \hat{\mathbf{z}}$ being constant can then be taken outside of the integral (i.e. integrate each component separately).

9) Calculate $\mathbf{I} = \iint_S \hat{\mathbf{e}}_r\ \mathrm{d}S$, the integral of $\mathbf{f} = \hat{\mathbf{e}}_r$ over the surface of a sphere S.

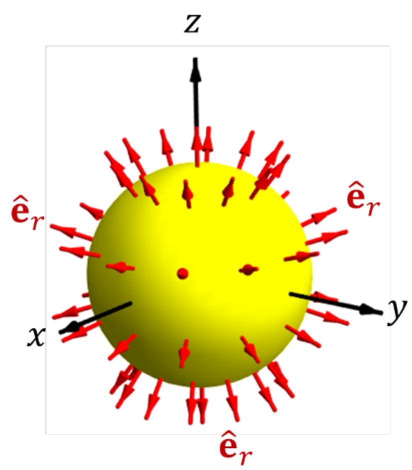
Solution:

In spherical coordinates we have: $\mathrm{d}S = R^2 \sin\theta\, \mathrm{d}\phi\, \mathrm{d}\theta$, with the parameters $\phi \in [0,2\pi]$ and $\theta \in [0,\pi]$. The unit vector $\hat{\mathbf{e}}_r$ varies throughout the surface, so we must write it in terms of constant rectangular unit vectors (look up the equation): $\hat{\mathbf{e}}_r = (\sin\theta\cos\phi)\hat{\mathbf{x}} + (\sin\theta\sin\phi)\hat{\mathbf{y}} + (\cos\theta)\hat{\mathbf{z}}$. So:

$$\mathbf{I} = \iint_S \hat{\mathbf{e}}_r\ \mathrm{d}S = \iint_S (\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})\ R^2 \sin\theta\, \mathrm{d}\phi\, \mathrm{d}\theta$$

$$= \hat{\mathbf{x}} \iint_S R^2 \sin^2\theta\cos\phi\, \mathrm{d}\phi\, \mathrm{d}\theta + \hat{\mathbf{y}} \iint_S R^2 \sin^2\theta\sin\phi\, \mathrm{d}\phi\, \mathrm{d}\theta + \hat{\mathbf{z}} \iint_S R^2 \sin\theta\cos\theta\, \mathrm{d}\phi\, \mathrm{d}\theta = \mathbf{0}$$

Each of these integrals, evaluated over the sphere, gives 0. The net result is zero as expected, because the vector sum of $\hat{\mathbf{e}}_r$ over a whole sphere will clearly cancel out: every element of the surface has an opposite one with exactly the opposite vector.



D. FLUX OF A VECTOR FIELD OVER A SURFACE

We define the flux of a vector field $\mathbf{F}(x, y, z)$ over a surface S as:

$$\Phi = \iint_S \underbrace{(\mathbf{F} \cdot \hat{\mathbf{n}})}_{\text{scalar}} \, \mathrm{d}S = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

$$\mathrm{d}\mathbf{S} \stackrel{\text{def}}{=} \hat{\mathbf{n}} \, \mathrm{d}S$$

Please note that $\hat{\mathbf{n}} = \mathbf{N}/||\mathbf{N}||$ is a **unit** vector normal to the surface, and the dot product $(\mathbf{F} \cdot \hat{\mathbf{n}})$ is giving us a scalar function $F_n(u, v)$ which we then integrate over the surface as explained earlier. The scalar function $F_n(u, v)$ gives us the component of the vector $\mathbf{F}$ in the direction $\hat{\mathbf{n}}$ perpendicular to the surface.

Suggestion when doing these problems: take it step by step. **First focus on obtaining the scalar integrand** $(\mathbf{F} \cdot \hat{\mathbf{n}})\mathrm{d}S$. Make sure it is a **scalar** function, and make sure that everything is expressed in terms of the parameters of the surface (plus constants) $(\mathbf{F} \cdot \hat{\mathbf{n}})\mathrm{d}S = f(u, v) \, \mathrm{d}u \, \mathrm{d}v$.

In many problems, it is easier to use the following expression for $\mathrm{d}\mathbf{S}$ in terms of $\mathbf{N}$ without having to normalize it, because the $||\mathbf{N}||$ dividing $\hat{\mathbf{n}}$ cancels out with the $||\mathbf{N}||$ multiplying in $\mathrm{d}S$:

$$\mathrm{d}\mathbf{S} \stackrel{\text{def}}{=} \hat{\mathbf{n}} \, \mathrm{d}S = \underbrace{\frac{\mathbf{N}}{||\mathbf{N}||}}_{\hat{\mathbf{n}}} \underbrace{||\mathbf{N}|| \mathrm{d}u \, \mathrm{d}v}_{\mathrm{d}S} = \mathbf{N} \, \mathrm{d}u \, \mathrm{d}v$$

INTUITIVE UNDERSTANDING:

This type of integral is extremely common in physics when the vector field $\mathbf{F}$ represents the flux density of something, so the flux integral represents the total flow. Some examples are given below.

Conserved quantity	**Flow of conserved quantity per unit area**	**Equation** (ρ = density of quantity; $\mathbf{v}$ = velocity of quantity)
Mass	Mass flux density $\mathbf{J}_m$ ($\mathrm{kg\,s^{-1}m^{-2}}$)	$\mathbf{J}_m = \rho \, \mathbf{v}$
Charge	Current density $\mathbf{J}_c$ ($\mathrm{C\,s^{-1}m^{-2}} \equiv \mathrm{A\,m^{-2}}$)	$\mathbf{J}_c = \rho \, \mathbf{v}$
Energy	Energy flux density ($\mathrm{J\,s^{-1}m^{-2}} \equiv \mathrm{W\,m^{-2}}$) or Power density	Different equations for different types of energy flux

With these flux density vectors, the total flux of the quantity through a given surface per unit time is:

$$I = \iint_S \underbrace{(\mathbf{J} \cdot \hat{\mathbf{n}})}_{\text{scalar}} \, \mathrm{d}S = \iint_S \mathbf{J}_c \cdot \mathrm{d}\mathbf{S}$$

10) Water is flowing through a cylindrical pipe of radius R. Let's assume that water flows fastest at the centre of the pipe with a speed v_0 (ms^{-1}) and slows down to zero at the walls of the tube in a linear way.

The "mass flux density" of water in this tube can then be approximated by $\mathbf{F} = \hat{\mathbf{z}}\, K v_0 (1 - \rho/R)$ where K $(\mathrm{kg\ m}^{-3})$ is the density of water, and ρ is the radial cylindrical coordinate. Calculate the total flux of water through the pipe per second.

1. **Find a parametrization of the surface**

The surface can be parametrised as a flat cylindrical plate in cylindrical coordinates $\phi \in [0,2\pi]$ and $\rho \in [0, R]$.

$$\mathbf{r}(\rho, \phi) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ 0 \end{pmatrix}$$

2. **Find the vector differential surface element $\mathrm{d}\mathbf{S}$**

The surface element for polar coordinates is $\mathrm{d}S = \rho\ \mathrm{d}\phi\ \mathrm{d}\rho$ and the normal to the surface is $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Hence, we know that $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\ \mathrm{d}S = \rho\ \mathrm{d}\phi\ \mathrm{d}\rho\ \hat{\mathbf{z}}$

If this was not obvious, you can follow the recipe:

$$\rightarrow \mathbf{N} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial \rho} & \frac{\partial x}{\partial \rho} \\ \frac{\partial z}{\partial \phi} & \frac{\partial x}{\partial \phi} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{vmatrix}$$

$$= \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial \rho} & 0 \\ \frac{\partial y}{\partial \phi} & 0 \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} 0 & \frac{\partial x}{\partial \rho} \\ 0 & \frac{\partial x}{\partial \phi} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix} = \hat{\mathbf{z}}\,\rho$$

Therefore, the surface element is $\mathrm{d}\mathbf{S} = \mathbf{N}\ \mathrm{d}\phi\ \mathrm{d}\rho = \hat{\mathbf{z}}\,\rho\ \mathrm{d}\phi\ \mathrm{d}\rho$

Also, $\mathrm{d}\mathbf{S} = \|\mathbf{N}\|\ \mathrm{d}\phi\ \mathrm{d}\rho = \rho\ \mathrm{d}\phi\ \mathrm{d}\rho$; and $\hat{\mathbf{n}} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \hat{\mathbf{z}}$

3. **Find the scalar integrand for the flux by doing the dot product $\mathbf{F} \cdot \mathrm{d}\mathbf{S}$**

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = \left(\hat{\mathbf{z}}\, K v_0 \left(1 - \frac{\rho}{R}\right)\right) \cdot (\rho\ \mathrm{d}\phi\ \mathrm{d}\rho\ \hat{\mathbf{z}}) = K v_0 \rho \left(1 - \frac{\rho}{R}\right)\ \mathrm{d}\phi\ \mathrm{d}\rho$$

4. **Calculate the flux integral**

The integral is now a scalar double integral.

$$\Phi = \iint_S \underbrace{\mathbf{F} \cdot \mathrm{d}\mathbf{S}}_{\text{scalar d}\phi} = \iint_S K v_0 \rho \left(1 - \frac{\rho}{R}\right)\ \mathrm{d}\phi\ \mathrm{d}\rho = K v_0 \int_0^{2\pi} \mathrm{d}\phi \int_0^R \left(\rho \left(1 - \frac{\rho}{R}\right)\right) \mathrm{d}\rho$$

$$= 2\pi K v_0 \left(\frac{\rho^2}{2} - \frac{\rho^3}{3R}\right)_{\rho=0}^{\rho=R} = 2\pi K v_0 R^2 \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}\pi K v_o R^2$$

That would be the flux of water in $(\mathrm{kg\ s}^{-1})$.

11) Find the flux of the field $\mathbf{F} = yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ through the outer surface of the cylinder with radius R and z-values $0 < z < h$ (consider only the curved surface, not the caps).

Solution:

$$\Phi = \iint_S \underbrace{(\mathbf{F} \cdot \hat{\mathbf{n}})}_{\text{scalar}} \, \mathrm{d}S = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

1. **Decide the parametrization of the surface:**

We are going to use cylindrical coordinates to integrate the cylindrical surface, by varying $\phi \in [0,2\pi]$ and $z \in [0, h]$.

$$\mathbf{r}(\phi, z) = \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ z \end{pmatrix}$$

2. **Find the vector surface differential** $\mathrm{d}\mathbf{S}$:

By geometric intuition, or by looking it up, we know that the scalar surface element for the curved surface of a cylinder is $\mathrm{d}S = R\,\mathrm{d}\phi\,\mathrm{d}z$. Also, by geometrical intuition we can identify the unit vector normal to the surface $\hat{\mathbf{n}} = \hat{\mathbf{e}}_\rho$.

So, by intuition, or looking it up, we can arrive at the vector surface differential:

$$\mathrm{d}\mathbf{S} = R\hat{\mathbf{e}}_\rho \mathrm{d}\phi\,\mathrm{d}z = (R\cos\phi\,\hat{\mathbf{x}} + R\sin\phi\,\hat{\mathbf{y}})\,\mathrm{d}\phi\,\mathrm{d}z$$

We can derive all the above from calculation of $\mathbf{N}$**,** as follows:

$$\mathbf{N} = \hat{\mathbf{x}}\begin{vmatrix} \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \\ \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} + \hat{\mathbf{y}}\begin{vmatrix} \frac{\partial z}{\partial\phi} & \frac{\partial x}{\partial\phi} \\ \frac{\partial z}{\partial z} & \frac{\partial x}{\partial z} \end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix} \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \end{vmatrix}$$

$$= \hat{\mathbf{x}}\begin{vmatrix} R\cos\phi & 0 \\ 0 & 1 \end{vmatrix} + \hat{\mathbf{y}}\begin{vmatrix} 0 & -R\sin\phi \\ 1 & 0 \end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix} \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} \\ 0 & 0 \end{vmatrix} = R\cos\phi\,\hat{\mathbf{x}} + R\sin\phi\,\hat{\mathbf{y}}$$

$$= R\hat{\mathbf{e}}_\rho$$

So that: $\mathrm{d}\mathbf{S} = \mathbf{N}\,\mathrm{d}\phi\,\mathrm{d}z = R\hat{\mathbf{e}}_\rho\,\mathrm{d}\phi\,\mathrm{d}z$

3. **Find the scalar integrand for the flux by doing the dot product** $\mathbf{F} \cdot \mathrm{d}\mathbf{S}$

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}) \cdot (R\cos\phi\,\hat{\mathbf{x}} + R\sin\phi\,\hat{\mathbf{y}})\,\mathrm{d}\phi\,\mathrm{d}z$$

we need to write the rectangular coordinates in terms of the coordinates of the surface: $\{x = R\cos\phi\,, y = R\sin\phi\,, z = z\}$ (Note that we used R and not ρ because our surface is at $\rho = R$):

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (zR\sin\phi\,\hat{\mathbf{x}} + zR\cos\phi\,\hat{\mathbf{y}} + R^2\cos\phi\sin\phi\,\hat{\mathbf{z}}) \cdot (R\cos\phi\,\hat{\mathbf{x}} + R\sin\phi\,\hat{\mathbf{y}})\,\mathrm{d}\phi\,\mathrm{d}z$$

$$= (zR^2\sin\phi\cos\phi + zR^2\sin\phi\cos\phi)\,\mathrm{d}\phi\,\mathrm{d}z = 2zR^2\sin\phi\cos\phi\;\mathrm{d}\phi\,\mathrm{d}z$$

4. **Calculate the (scalar) double integral**

$$\Phi = \iint_S \underbrace{(\mathbf{F}\cdot \mathrm{d}\mathbf{S})}_{\text{scalar } \mathrm{d}\phi} = \iint_S 2zR^2 \sin\phi\cos\phi \,\mathrm{d}\phi\,\mathrm{d}z = 2R^2 \int_0^h z\,\mathrm{d}z \int_0^{2\pi} \sin\phi\cos\phi\,\mathrm{d}\phi$$

$$= 2R^2 \left(\frac{h^2}{2}\right)\left(\frac{1}{2}\sin^2\phi\right)_{\phi=0}^{\phi=2\pi} = 0$$

Alternatively, one can obtain the integrand as $(\mathbf{F}\cdot\hat{\mathbf{n}})\,\mathrm{d}S$ by calculating $\hat{\mathbf{n}}$ and $\mathrm{d}S$

$$\mathrm{d}S = \|\mathbf{N}\|\mathrm{d}\phi\,\mathrm{d}z = \sqrt{(R\cos\phi)^2 + (R\sin\phi)^2} = R\,\mathrm{d}\phi\,\mathrm{d}z$$

$$\hat{\mathbf{n}} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{R\hat{\mathbf{e}}_\rho}{R} = \hat{\mathbf{e}}_\rho$$

$$\begin{aligned}\mathbf{F}\cdot\hat{\mathbf{n}} = (yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}})\cdot\hat{\mathbf{e}}_\rho &= (yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}})\cdot(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})\\ &= (zR\sin\phi\,\hat{\mathbf{x}} + zR\cos\phi\,\hat{\mathbf{y}} + R^2\cos\phi\sin\phi\,\hat{\mathbf{z}})\cdot(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})\\ &= 2zR\sin\phi\cos\phi\end{aligned}$$

So the integrand becomes:

$$(\mathbf{F}\cdot\hat{\mathbf{n}})\,\mathrm{d}S = (2zR\sin\phi\cos\phi)(R\,\mathrm{d}\phi\,\mathrm{d}z) = 2zR^2\sin\phi\cos\phi\,\mathrm{d}\phi\,\mathrm{d}z$$

Exactly as above. This procedure is usually longer, because we need to calculate $\|\mathbf{N}\|$ and $\hat{\mathbf{n}}$, and later multiply times the scalar $\mathrm{d}S = \|\mathbf{N}\|\mathrm{d}\phi\,\mathrm{d}z$ such that the $\|\mathbf{N}\|$ cancels out.

PROBLEMS

SURFACE INTEGRALS OF SCALAR INTEGRANDS (INCLUDING CALCULATION OF AREAS)

12) Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 1$.

Solution:

$$A = \iint_S \mathrm{d}S$$

1. **Decide the parametrization of the surface** S. There are many options for this! Let's use x and y as the two parameters to be integrated in the unit disk in the xy plane. Hence, the surface parametrization is:

$$\mathbf{r}(x,y) = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}$$

2. **Find the surface element** $\mathrm{d}S$ as a function of the parameter differentials $\mathrm{d}x\ \mathrm{d}y$

Let's start by calculating $\mathbf{N}$:

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial x}{\partial x} \\ \frac{\partial z}{\partial y} & \frac{\partial x}{\partial y} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} 0 & 2x \\ 1 & 2y \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} 2x & 1 \\ 2y & 0 \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= -2x\, \hat{\mathbf{x}} - 2y\, \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

Therefore:

$$\mathrm{d}S = \|\mathbf{N}\|\ \mathrm{d}x\ \mathrm{d}y = \sqrt{(2x)^2 + (2y)^2 + 1^2}\ \mathrm{d}x\ \mathrm{d}y = \sqrt{4(x^2 + y^2) + 1}\ \mathrm{d}x\ \mathrm{d}y$$

3. Find the integrand in terms of the parameters: $1\ \mathrm{d}S(x,y)$ is ready for integration.

4. **Perform the integration** over the xy parameter space (the region is the unit disk, obtained by substituting $z = 1$, so it needs to be an iterated integral with variable limits).

$$\mathrm{A} = \iint_S \mathrm{d}S = \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{4(x^2 + y^2) + 1}\ \mathrm{d}x \right) \mathrm{d}y$$

This might be solvable with care and labour, but it seems easier to change the double integration into polar coordinates by using a change of variables, $x = \rho\cos\phi$, $y = \rho\sin\phi$ and don't forget the Jacobian $\mathrm{d}x\ \mathrm{d}y = \rho\ \mathrm{d}\rho\ \mathrm{d}\phi$. That way the limits of integration become constant and the integral is separable:

$$\mathrm{A} = \int_0^{2\pi} \left(\int_0^1 \rho\sqrt{4\rho^2 + 1}\ \mathrm{d}\rho \right) \mathrm{d}\phi = (\text{separation}) = \int_0^{2\pi} \mathrm{d}\phi \int_0^1 \rho(4\rho^2 + 1)^{\frac{1}{2}}\ \mathrm{d}\rho$$
$$= 2\pi \left[\frac{1}{\left(\frac{3}{2}\right)(4)(2)} (4\rho^2 + 1)^{\frac{3}{2}} \right]_0^1 = \frac{2\pi}{12}\left(5^{\frac{3}{2}} - 1\right) = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

13) Find the integral $\iint_S x^2 \, \mathrm{d}S$ where S is the sphere of unit radius centred in the origin.

Solution:

1. **Decide the parametrization** of the surface S. Clearly this integration is well suited for spherical coordinates with $r = 1$:

$\mathbf{r}(\theta, \phi) = (\sin\theta\cos\phi\,, \sin\theta\sin\phi\,, a\cos\theta)$ with parameters $\theta \in [0, \pi]$ and $\phi \in [0,2\pi]$.

2. **Find the surface differential** $\mathrm{d}S$ in terms of $\mathrm{d}\theta\ \mathrm{d}\phi$. These are spherical coordinates, so we look it up:

$$\mathrm{d}S = r^2 \sin\theta\, \mathrm{d}\theta\ \mathrm{d}\phi = \sin\theta\, \mathrm{d}\theta\ \mathrm{d}\phi \text{ when } r = 1$$

3. **Find the integrand in terms of the parameters** (simply substitute $x = \sin\theta\cos\phi$)

$$x^2 = \sin^2\theta\cos^2\phi$$

4. **Calculate the integral**.

$$I = \iint_S x^2 \, \mathrm{d}S = \int_0^{2\pi}\int_0^{\pi} \underbrace{\sin^2\theta\cos^2\phi}_{x^2}\, \underbrace{\sin\theta\, \mathrm{d}\theta\ \mathrm{d}\phi}_{\mathrm{d}S} = (\text{separation}) = \int_0^{2\pi} \cos^2\phi\, \mathrm{d}\phi \int_0^{\pi} \sin^3\theta\, \mathrm{d}\theta$$

Let's perform the integrals separately:

$$\int_0^{2\pi} \cos^2\phi\, \mathrm{d}\phi = \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2\phi\right) \mathrm{d}\phi = \left[\frac{1}{2}\phi + \frac{1}{4}\sin 2\phi\right]_0^{2\pi} = \pi$$

$$\int_0^{\pi} \sin^3\theta\, \mathrm{d}\theta = \int_0^{\pi} \sin\theta\sin^2\theta\, \mathrm{d}\theta = \int_0^{\pi} \sin\theta\,(1 - \cos^2\theta)\mathrm{d}\theta = \int_0^{\pi} (\sin\theta - \sin\theta\cos^2\theta)\mathrm{d}\theta$$
$$= \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$$

Therefore,

$$I = \int_0^{2\pi} \cos^2\phi\, \mathrm{d}\phi \int_0^{\pi} \sin^3\theta\, \mathrm{d}\theta = (\pi)\left(\frac{4}{3}\right) = \frac{4\pi}{3}$$

14) Find the area of the surface $x^2 - 2y - 2z = 0$ that lies above the triangle bounded by the lines $x = 2$, $y = 0$ and $y = 3x$ in the xy plane.

$$A = \iint_S \mathrm{d}S$$

1. **Decide the parametrization of the surface** S

This is not any easily visualized surface, so we can just take it as a function $z(x, y) = \frac{1}{2}x^2 - y$ and use x and y as the two parameters to be integrated in the given triangle in the xy plane. Hence, the surface parametrization is:

$$\mathbf{r}(x, y) = \begin{pmatrix} x \\ y \\ \frac{1}{2}x^2 - y \end{pmatrix}$$

2. **Find the element of surface** $\mathrm{d}S$ as a function of the parameter differentials $\mathrm{d}x\ \mathrm{d}y$

Let's start by calculating $\mathbf{N}$:

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial x}{\partial x} \\ \frac{\partial z}{\partial y} & \frac{\partial x}{\partial y} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} 0 & x \\ 1 & -1 \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} x & 1 \\ -1 & 0 \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= -x\,\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

Therefore:

$$\mathrm{d}S = \|\mathbf{N}\|\ \mathrm{d}x\ \mathrm{d}y = \sqrt{x^2 + 1^2 + 1^2}\ \mathrm{d}x\ \mathrm{d}y = \sqrt{x^2 + 2}\ \mathrm{d}x\ \mathrm{d}y$$

3. **Perform the integration** over the xy parameter space (iterated integral with variable limits). I choose to integrate y first, so the limits will be 0 to $3x$. Then x will be integrated from 0 to 2.

$$A = \iint_S \mathrm{d}S = \int_0^2 \int_0^{3x} \underbrace{\sqrt{x^2 + 2}\ \mathrm{d}y\ \mathrm{d}x}_{\mathrm{d}S} = \int_0^2 \left(\int_0^{3x} (x^2 + 2)^{1/2}\ \mathrm{d}y \right) \mathrm{d}x$$

We cannot do separation of the integrals because the limits are not constants, we need to do the iterated integrals:

$$A = \int_0^2 \left[y(x^2 + 2)^{\frac{1}{2}} \right]_{y=0}^{y=3x} \mathrm{d}x = \int_0^2 3x(x^2 + 2)^{\frac{1}{2}}\ \mathrm{d}x = \left[(x^2 + 2)^{\frac{3}{2}} \right]_0^2 = 6\sqrt{6}$$

SURFACE INTEGRALS OF VECTOR INTEGRANDS

15) Find the centre of mass of a hemispherical dome (the hemispherical shell $x^2 + y^2 + z^2 = a^2$, $z \geq 0$) with constant density

Solution:

The centre of mass (given that the density is constant) can be calculated as the average of the position vector in the entire surface:

$$\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \frac{\iint_S \mathbf{r}\ \mathrm{d}S}{\iint_S \mathrm{d}S}$$

1. **Decide the parametrization of the surface** S.

Clearly this integration is well suited for spherical coordinates:

$\mathbf{r}(\theta, \phi) = (a \sin\theta \cos\phi\,, a \sin\theta \sin\phi\,, a\cos\theta)^T$ with parameters $\theta \in [0, \frac{\pi}{2}]$ and $\phi \in [0, 2\pi]$.

2. **Find the differential of surface** in terms of $\mathrm{d}\theta\ \mathrm{d}\phi$. These are spherical coordinates, so we look it up:

$$\mathrm{d}S = a^2 \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi$$

3. **Calculate the denominator integral** (the total surface of the hemisphere)

$$\iint_S \mathrm{d}S = \int_0^{2\pi}\int_0^{\pi/2} a^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi = (\text{separation}) = a^2\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\sin\theta\,\mathrm{d}\theta = 2\pi a^2[-\cos\theta]_0^{\pi/2}$$
$$= 2\pi a^2$$

4. **Calculate the numerator integral**. It is a vector, so apply linearity:

$$\iint_S \mathbf{r}\,\mathrm{d}S = \iint_S \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mathrm{d}S = \iint_S (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})\,\mathrm{d}S = \hat{\mathbf{x}}\iint_S x\,\mathrm{d}S + \hat{\mathbf{y}}\iint_S y\,\mathrm{d}S + \hat{\mathbf{z}}\iint_S z\,\mathrm{d}S$$

(i.e. calculate the integral component by component). Remember that the integrand containing x, y and z needs to be evaluated at the surface, so we need to write them in terms of the parametric surface, i.e. ($x = a\sin\theta\cos\phi\,, y = a\sin\theta\sin\phi\,, z = a\cos\theta$).

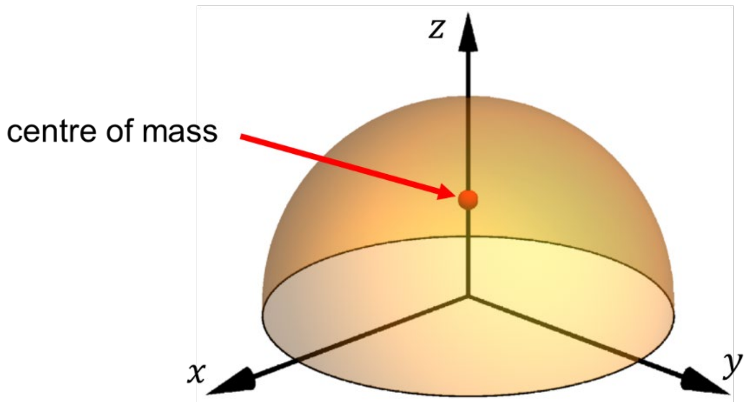
$$\iint_S x\,\mathrm{d}S = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \underbrace{a\sin\theta\cos\phi}_{x}\ \underbrace{a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}_{\mathrm{d}S} = a^2\underbrace{\int_0^{2\pi}\cos\phi\,\mathrm{d}\phi}_{0}\int_0^{\frac{\pi}{2}}\sin^2\theta\,\mathrm{d}\theta = 0$$

$$\iint_S y\,\mathrm{d}S = \int_0^{2\pi}\int_0^{\pi/2} \underbrace{a\sin\theta\sin\phi}_{y}\ \underbrace{a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}_{\mathrm{d}S} = a^2\underbrace{\int_0^{2\pi}\sin\phi\,\mathrm{d}\phi}_{0}\int_0^{\pi/2}\sin^2\theta\,\mathrm{d}\theta = 0$$

$$\iint_S z\,\mathrm{d}S = \int_0^{2\pi}\int_0^{\pi/2} \underbrace{a\cos\theta}_{z}\ \underbrace{a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}_{\mathrm{d}S} = a^3\underbrace{\int_0^{2\pi}\mathrm{d}\phi}_{2\pi}\underbrace{\int_0^{\pi/2}\cos\theta\sin\theta\,\mathrm{d}\theta}_{\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}=\frac{1}{2}} = a^3\pi$$

So that we can finally substitute into the equation for the centre of mass:

$$\mathbf{r}_0 = \frac{\iint_S \mathbf{r}\,\mathrm{d}S}{\iint_S \mathrm{d}S} = \frac{0\hat{\mathbf{x}} + 0\hat{\mathbf{x}} + a^3\pi\hat{\mathbf{z}}}{2\pi a^2} = \frac{a}{2}\hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ a/2 \end{pmatrix}$$



CALCULATION OF FLUX:

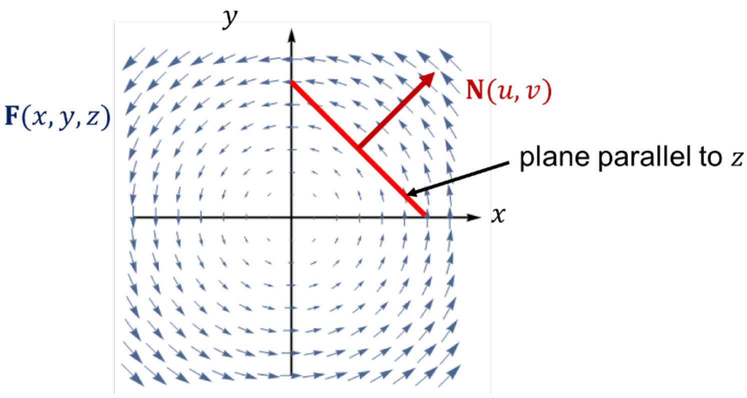
16) Calculate the flux $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ of the vector field $\mathbf{F} = x\,\hat{\mathbf{y}} - y\,\hat{\mathbf{x}}$ across the surface $\mathbf{r}(u, v) = (u, 1 - u, v)$ for $u \in [0,1]$ and $v \in [0,1]$. Define flux as outward from the origin.

Solution:

1. **Parametrize the surface**. Already provided in the question

$\mathbf{r}(u, v) = \begin{pmatrix} u \\ 1 - u \\ v \end{pmatrix}$ with $u \in [0,1]$ and $v \in [0,1]$ (first octant)

This is a square-shaped region of a plane parallel to z and diagonal in xy. Looking from above:



2. **Find the surface differential**.

Start by calculating the normal vector:

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \hat{\mathbf{x}} \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = -\hat{\mathbf{x}} - \hat{\mathbf{y}}$$

This unit vector is pointing inward, but the question specifically requests calculating the flux "outward" from the origin. The normal to a surface has an arbitrary sign, so we can just change the sign:

$$\mathbf{N} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$$

Therefore

$$\mathrm{d}\mathbf{S} = \mathbf{N}\,\mathrm{d}u\,\mathrm{d}v = (\hat{\mathbf{x}} + \hat{\mathbf{y}})\,\mathrm{d}u\,\mathrm{d}v$$

3. **Find the integrand** (it must be a scalar differential, the dot product of $\mathbf{F} \cdot \mathrm{d}\mathbf{S}$)

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (x\,\hat{\mathbf{y}} - y\,\hat{\mathbf{x}}) \cdot (\hat{\mathbf{x}} + \hat{\mathbf{y}})\,\mathrm{d}u\,\mathrm{d}v = (-y + x)\,\mathrm{d}u\,\mathrm{d}v$$

The integrand needs to be evaluated at the surface, so we substitute the parametric equation of the surface $x = u$ and $y = 1 - u$

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (2u - 1)\mathrm{d}u\,\mathrm{d}v$$

Which is in the form we need it. A scalar differential function of the parameters which can now be integrated.

4. **Calculate the flux scalar integral**

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_0^1 \int_0^1 (2u-1)\mathrm{d}u\, \mathrm{d}v = \int_0^1 \mathrm{d}v \int_0^1 (2u-1)\mathrm{d}u = [u^2-u]_0^1 = 0$$

There is no net flux. i.e. as much field is flowing in one direction as in the other.

This was easy to guess from the diagram!

17) Calculate the flux $\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}$ over the outward surface of the unit sphere centred at the origin, where the vector field is $\mathbf{F}(\mathbf{r}) = \mathbf{r}$.

Solution:

1. **Find the parametrization of the surface**:

We use spherical coordinates for the surface: $\mathbf{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$ with $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$.

2. **Find the surface element**:

$$\begin{aligned}
\mathbf{N} = \hat{\mathbf{x}}\begin{vmatrix} \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta} \\ \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi}\end{vmatrix} + \hat{\mathbf{y}}\begin{vmatrix} \frac{\partial z}{\partial\theta} & \frac{\partial x}{\partial\theta} \\ \frac{\partial z}{\partial\phi} & \frac{\partial x}{\partial\phi}\end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} \\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi}\end{vmatrix} \\
&= \hat{\mathbf{x}}\begin{vmatrix} \cos\theta\sin\phi & -\sin\theta \\ \sin\theta\cos\phi & 0\end{vmatrix} + \hat{\mathbf{y}}\begin{vmatrix} -\sin\theta & \cos\theta\cos\phi \\ 0 & -\sin\theta\sin\phi\end{vmatrix} \\
&+ \hat{\mathbf{z}}\begin{vmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ -\sin\theta\sin\phi & \sin\theta\cos\phi\end{vmatrix} \\
&= \hat{\mathbf{x}}(\sin^2\theta\cos\phi) + \hat{\mathbf{y}}(\sin^2\theta\sin\phi) + \hat{\mathbf{z}}\big(\cos\theta\sin\theta\,(\cos^2\phi + \sin^2\phi)\big) \\
&= \hat{\mathbf{x}}(\sin^2\theta\cos\phi) + \hat{\mathbf{y}}(\sin^2\theta\sin\phi) + \hat{\mathbf{z}}(\cos\theta\sin\theta) \\
&= \big(\hat{\mathbf{x}}(\sin\theta\cos\phi) + \hat{\mathbf{y}}(\sin\theta\sin\phi) + \hat{\mathbf{z}}(\cos\theta)\big)\sin\theta
\end{aligned}$$

Therefore:

$$\mathrm{d}\mathbf{S} = \mathbf{N}\,\mathrm{d}\theta\,\mathrm{d}\phi = \big(\hat{\mathbf{x}}(\sin\theta\cos\phi) + \hat{\mathbf{y}}(\sin\theta\sin\phi) + \hat{\mathbf{z}}(\cos\theta)\big)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

3. **Find the scalar integrand** (**perform the dot product $\mathbf{F}\cdot \mathrm{d}\mathbf{S}$**) in terms of the parameters.

$$\mathbf{F}\cdot \mathrm{d}\mathbf{S} = (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})\cdot\Big(\big(\hat{\mathbf{x}}(\sin\theta\cos\phi) + \hat{\mathbf{y}}(\sin\theta\sin\phi) + \hat{\mathbf{z}}(\cos\theta)\big)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\Big)$$

We need to evaluate the vector field at the surface, so we need to substitute the parametric equation for the surface into x, y, z:

$$\begin{aligned}
\mathbf{F}\cdot \mathrm{d}\mathbf{S} = (\sin\theta\cos\phi\ \hat{\mathbf{x}} + \sin\theta\sin\phi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}) \\
&\cdot\Big(\big(\hat{\mathbf{x}}(\sin\theta\cos\phi) + \hat{\mathbf{y}}(\sin\theta\sin\phi) + \hat{\mathbf{z}}(\cos\theta)\big)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\Big) \\
&= (\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= (\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi
\end{aligned}$$

4. **Perform the scalar double integration:**

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta = 2\pi[-\cos\theta]_0^{\pi} = 2\pi(--1--1) = 4\pi$$

Easier methods:

A. To simplify step 3, you could notice that the unit radial vector is present in $\mathbf{N} = \big(\hat{\mathbf{x}}(\sin\theta\cos\phi) + \hat{\mathbf{y}}(\sin\theta\sin\phi) + \hat{\mathbf{z}}(\cos\theta)\big)\sin\theta = \hat{\mathbf{e}}_r\sin\theta$
Such that $\mathrm{d}\mathbf{S} = \mathbf{N}\,\mathrm{d}\theta\,\mathrm{d}\phi = \hat{\mathbf{e}}_r\sin\theta\;\mathrm{d}\theta\,\mathrm{d}\phi$,
Such that $\mathbf{F}\cdot\mathrm{d}\mathbf{S} = (\mathbf{r})\cdot(\hat{\mathbf{e}}_r\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi) = r\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, which evaluated at $r=1$ gives the correct integrand, so you can directly go to step 4 above.

B. To simplify **all steps**, you could build $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S$ as follows.
The normal to the surface of a sphere is $\hat{\mathbf{n}} = \hat{\mathbf{e}}_r$,
The vector field is $\mathbf{F} = \mathbf{r} = r\hat{\mathbf{e}}_r$
The integrand is $\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = r\,\mathrm{d}S$, which evaluated at $r=1$ is simply $\mathrm{d}S$
Hence, $\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_S \mathrm{d}S = (\text{surface of unit sphere}) = 4\pi$

18) Calculate the flux $\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ of the vector field $\mathbf{F} = z\,\hat{\mathbf{z}}$ across the portion of the sphere $x^2+y^2+z^2=a^2$ in the first octant ($x,y,z\geq 0$). Define the flux as outward (away from the origin).

Solution:

1. **Parametrize the surface**. Clearly this surface is perfectly suited for spherical coordinates

$$\mathbf{r}(\theta,\phi) = \begin{pmatrix} a\sin\theta\cos\phi \\ a\sin\theta\sin\phi \\ a\cos\theta \end{pmatrix} \text{ with } \theta\in[0,\tfrac{\pi}{2}] \text{ and } \phi\in[0,\pi/2] \text{ (first octant)}$$

2. **Find the differential surface**.

Follow steps in previous problem with $r=a$, or you can look it up in the table of differentials provided in the exam. Differential for spherical coordinates in the radial direction is:

$$\mathrm{d}\mathbf{S} = a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\hat{\mathbf{e}}_r \text{ (pointing outward)}$$

3. **Find the integrand** (it must be a scalar differential, the dot product of $\mathbf{F}\cdot\mathrm{d}\mathbf{S}$)

$$\mathbf{F}\cdot\mathrm{d}\mathbf{S} = (z\,\hat{\mathbf{z}})\cdot(a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\hat{\mathbf{e}}_r)$$

We need to find this integrand as a scalar function of the parameters only $f(\theta,\phi)\mathrm{d}\phi\,\mathrm{d}\theta$. Therefore, we need to substitute $z = a\cos\theta$.

$$\mathbf{F}\cdot\mathrm{d}\mathbf{S} = (a\cos\theta\;\hat{\mathbf{z}})\cdot(a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\hat{\mathbf{e}}_r)$$

We also need to evaluate the dot product $\hat{\mathbf{z}}\cdot\hat{\mathbf{e}}_r$. One method is to look up the unit vector in the spherical radial coordinate $\hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}$. Another method is to think it geometrically. In any case: $\hat{\mathbf{z}}\cdot\hat{\mathbf{e}}_r = \cos\theta$. So that:

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (a \cos\theta)(a^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta)(\cos\theta) = a^3 \cos^2\theta \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta$$

Which is in the form we need it. A scalar function of the two parameters which can now be integrated.

4. **Calculate the flux scalar integral**

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^{\pi/2} \int_0^{\pi/2} a^3 \cos^2\theta \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta = a^3 \int_0^{\pi/2} \mathrm{d}\phi \int_0^{\pi/2} \cos^2\theta \sin\theta \, \mathrm{d}\theta$$

$$= a^3 \left(\frac{\pi}{2}\right) \left[-\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}} = \frac{a^3\pi}{6}$$

19) Calculate the flux $\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}$ of the vector field $\mathbf{F} = -y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$ across the portion of the sphere $x^2+y^2+z^2=a^2$ in the first octant ($x,y,z\geq 0$). Define the flux as outward (away from the origin).

Solution:

1. **Parametrize the surface**. Clearly this surface is perfectly suited for spherical coordinates

$\mathbf{r}(\theta,\phi) = \begin{pmatrix} a\sin\theta\cos\phi \\ a\sin\theta\sin\phi \\ a\cos\theta \end{pmatrix}$ with $\theta \in [0,\frac{\pi}{2}]$ and $\phi\in[0,\pi/2]$ (first octant)

2. **Find the differential surface**.

We have done this in previous problems, or you can look it up in the table of differentials provided in the exam. Differential for spherical coordinates in the radial direction is:

$$\mathrm{d}\mathbf{S} = a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\hat{\mathbf{e}}_r \text{ (pointing outward)}$$

3. **Find the integrand** (it must be a scalar differential, the dot product of $\mathbf{F}\cdot\mathrm{d}\mathbf{S}$)

$$\mathbf{F}\cdot\mathrm{d}\mathbf{S} = (-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}})\cdot(a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\hat{\mathbf{e}}_r)$$

We need to find this integrand as a scalar function of the parameters only $f(\theta,\phi)\mathrm{d}\phi\,\mathrm{d}\theta$. Therefore, we need to substitute $x = a\sin\theta\cos\phi$ and $y = a\sin\theta\sin\phi$.

$$\mathbf{F}\cdot\mathrm{d}\mathbf{S} = (-a\sin\theta\sin\phi\ \hat{\mathbf{x}} + a\sin\theta\cos\phi\ \hat{\mathbf{y}})\cdot(a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\hat{\mathbf{e}}_r)$$

We also need to evaluate the dot product with $\hat{\mathbf{e}}_r$. We can look up the unit vector in the spherical radial coordinate $\hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}$.

$$\begin{aligned}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = {} & (-a\sin\theta\sin\phi\ \hat{\mathbf{x}} + a\sin\theta\cos\phi\ \hat{\mathbf{y}}) \\ & \cdot\big(a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}})\big)\end{aligned}$$

$$\begin{aligned} &= (-a\sin\theta\sin\phi)(a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta)(\sin\theta\cos\phi) + (a\sin\theta\cos\phi)(a^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta)(\sin\theta\sin\phi) \\ &= a^3\sin^3\theta\,(-\sin\phi\cos\phi + \sin\phi\cos\phi)\mathrm{d}\phi\,\mathrm{d}\theta = 0\,\mathrm{d}\phi\,\mathrm{d}\theta\end{aligned}$$

This would have been easy to find if we had parametrized the surface in rectangular coordinates:

$$\mathbf{r}(x,y) = \begin{pmatrix} x \\ y \\ \sqrt{x^2+y^2}\end{pmatrix}$$

$$\mathbf{N} = \hat{\mathbf{x}}\begin{vmatrix}\frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y}\end{vmatrix} + \hat{\mathbf{y}}\begin{vmatrix}\frac{\partial z}{\partial x} & \frac{\partial x}{\partial x} \\ \frac{\partial z}{\partial y} & \frac{\partial x}{\partial y}\end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix}\frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y}\end{vmatrix} = \hat{\mathbf{x}}\begin{vmatrix} 0 & \frac{x}{\sqrt{\cdot}} \\ 1 & \frac{y}{\sqrt{\cdot}}\end{vmatrix} + \hat{\mathbf{y}}\begin{vmatrix}\frac{x}{\sqrt{\cdot}} & 1 \\ \frac{y}{\sqrt{\cdot}} & 0\end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix}1 & 0 \\ 0 & 1\end{vmatrix}$$

$$= \frac{-\hat{\mathbf{x}}x - \hat{\mathbf{y}}y}{\sqrt{x^2+y^2}} + \hat{\mathbf{z}}$$

So that $\mathbf{F}\cdot\ \mathbf{N} = (yx - xy)/\sqrt{x^2+y^2} = 0$

4. **Calculate the flux scalar integral:**

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = 0$$

20) Calculate the flux of the vector field $\mathbf{F} = (x^2 z, xy^2, z)$ through the outer surface of $z = x^2 + y^2, 0 \leq z \leq 1, x \geq 0$ and $y \geq 0$.

Solution:

1. **Parametrise the surface**.

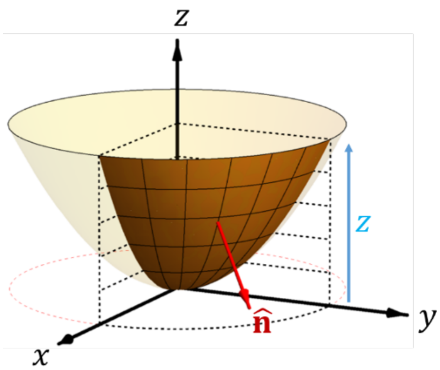
It is a paraboloid. There is no **single** right way of doing this, there are **many** ways.

Option 1: We could go for the parametrization (see later for a solution in this case):

$$\mathbf{r} = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix}$$

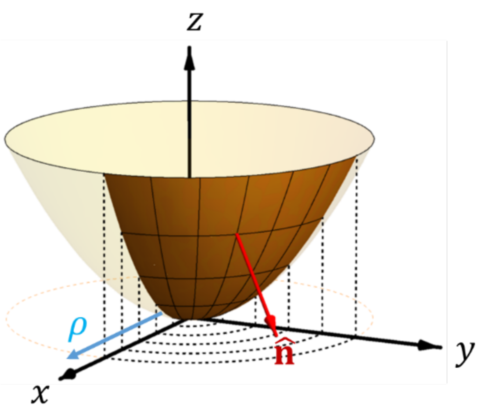
Option 2: The rotational symmetry of the surface suggests it would be easier if we used the polar angle $u = \phi$ as one parameter, and some other parameter that moves us up and down the parabola. For example, we could use z as parameter.

$$\mathbf{r} = \begin{pmatrix} \sqrt{z}\cos\phi \\ \sqrt{z}\sin\phi \\ z \end{pmatrix}$$



Option 3: Alternatively, we could use the radial coordinate $\rho = \sqrt{z}$ to move us up and down the parabola, which would allow us to remove the square roots:

$$\mathbf{r} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ \rho^2 \end{pmatrix}$$



Any of the above would be a valid parametrization of the surface. Let's use **Option 3**. The parameters must be integrated in the region $\rho \in [0,1]$ and $\phi = [0, \pi/2]$.

2. **Obtain the vector surface element** $\mathrm{d}\mathbf{S}$

In this case I consider it too risky to obtain $\mathrm{d}S$ by physical intuition. So, I directly use the definition of the vector $\mathbf{N}$ for this paraboloid.

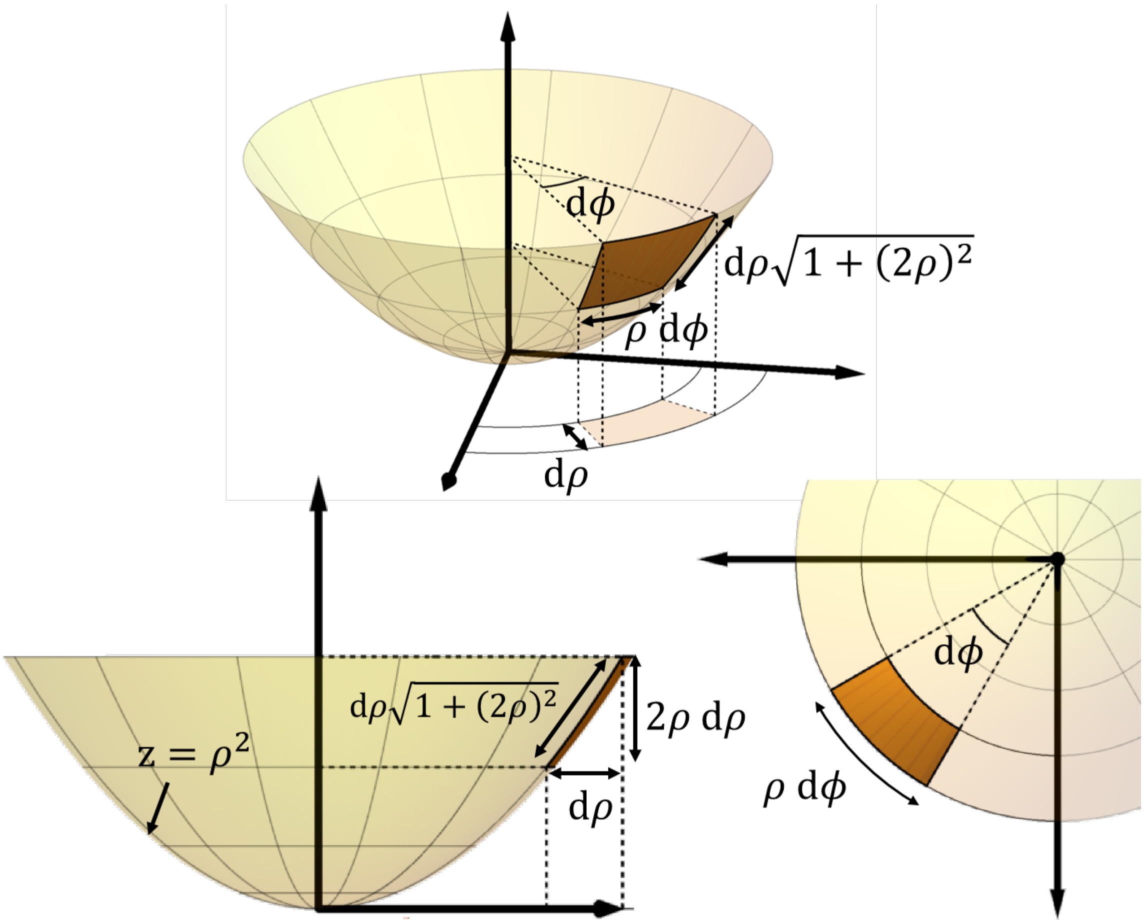
$$\mathbf{N} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial \phi} & \frac{\partial x}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial x}{\partial \rho} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \\ \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} \end{vmatrix}$$
$$= \hat{\mathbf{x}} \begin{vmatrix} \rho\cos\phi & 0 \\ \sin\phi & 2\rho \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} 0 & -\rho\sin\phi \\ 2\rho & \cos\phi \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} -\rho\sin\phi & \rho\cos\phi \\ \cos\phi & \sin\phi \end{vmatrix}$$
$$= \hat{\mathbf{x}} 2\rho^2 \cos\phi + \hat{\mathbf{y}} 2\rho^2 \sin\phi + \hat{\mathbf{z}}(-\rho\sin^2\phi - \rho\cos^2\phi) = 2\rho^2 \hat{\mathbf{e}}_\rho - \rho\hat{\mathbf{z}}$$

The surface vector differential element is therefore $\mathrm{d}\mathbf{S} = \mathbf{N}\,\mathrm{d}\rho\,\mathrm{d}\phi = \left(2\rho^2\hat{\mathbf{e}}_\rho - \rho\hat{\mathbf{z}}\right)\mathrm{d}\rho\,\mathrm{d}\phi$

For completeness, let's calculate here the scalar surface element $\mathrm{d}S$ and the normal vector, to see why this mathematical form for $\mathrm{d}\mathbf{S}$ makes sense geometrically:

$$\mathrm{d}S = \|\mathbf{N}\|\,\mathrm{d}\rho\,\mathrm{d}\phi = \sqrt{4\rho^4 + \rho^2}\,\mathrm{d}\rho\,\mathrm{d}\phi = \rho\sqrt{4\rho^2 + 1}\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$\hat{\mathbf{n}} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{2\rho^2\hat{\mathbf{e}}_\rho - \rho\hat{\mathbf{z}}}{\rho\sqrt{4\rho^2 + 1}}$$



3. **Find the scalar integrand (do the dot product $\mathbf{F}\cdot\mathrm{d}\mathbf{S}$)**. Remember that the integrand must be written in terms of the parameters.

So first we need to substitute (x, y, z) in terms of the parameters (ρ, ϕ) in the vector field:

$$\mathbf{F} = (x^2 z, xy^2, z) = (\rho^4 \cos^2\phi\,, \rho^3 \cos\phi \sin^2\phi\,, \rho^2)$$

Now we do the dot product:

$$\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \mathbf{F}\cdot\overbrace{\mathbf{N}\,\mathrm{d}\rho\,\mathrm{d}\phi}^{\mathrm{d}\mathbf{S}} = \begin{pmatrix}\rho^4\cos^2\phi \\ \rho^3\cos\phi\sin^2\phi \\ \rho^2\end{pmatrix}\cdot\begin{pmatrix}2\rho^2\cos\phi \\ 2\rho^2\sin\phi \\ -\rho\end{pmatrix}\mathrm{d}\rho\,\mathrm{d}\phi$$
$$= 2\rho^6\cos^3\phi + 2\rho^5\cos\phi\sin^3\phi - \rho^3$$

(Note that we could have done $\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ and get the same result. But doing it directly with $\mathbf{F}\cdot\mathrm{d}\mathbf{S}$ saved us from the ugly square roots needed to calculate $\|\mathbf{N}\|$ which eventually cancel out anyway because they appear dividing in $\hat{\mathbf{n}}$ but multiplying in $\mathrm{d}S$)

4. **Calculate the double scalar integral.**

$$\Phi = \iint_S \underbrace{\mathbf{F}\cdot\mathrm{d}\mathbf{S}}_{\text{scalar }\mathrm{d}\phi} = \iint_{S(\rho\phi)} (2\rho^6\cos^3\phi + 2\rho^5\cos\phi\sin^3\phi - \rho^3)\,\mathrm{d}\rho\,\mathrm{d}\phi$$

Linearity of the integration allows us to split this integral into three different double integrals, each of which can be split into the angular and the radial part by separation:

$$= \left(\int_0^{\frac{\pi}{2}}\cos^3\phi\,\mathrm{d}\phi\right)\left(\int_0^1 2\rho^6\,\mathrm{d}\rho\right) + \left(\int_0^{\frac{\pi}{2}}\cos\phi\sin^3\phi\,\mathrm{d}\phi\right)\left(\int_0^1 2\rho^5\,\mathrm{d}\rho\right) - \left(\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\right)\left(\int_0^1\rho^3\,\mathrm{d}\rho\right)$$

We can do each of these integrals:

$$\int_0^{\frac{\pi}{2}}\cos^3\phi\,\mathrm{d}\phi = \int_0^{\frac{\pi}{2}}\cos\phi\cos^2\phi\,\mathrm{d}\phi = \int_0^{\frac{\pi}{2}}\cos\phi\,(1-\sin^2\phi)\mathrm{d}\phi$$
$$= \int_0^{\frac{\pi}{2}}\cos\phi\,\mathrm{d}\phi - \int_0^{\frac{\pi}{2}}\cos\phi\sin^2\phi\,\mathrm{d}\phi = (\sin\phi)_{\phi=0}^{\phi=\frac{\pi}{2}} - \left(\frac{1}{3}\sin^3\phi\right)_{\phi=0}^{\phi=\frac{\pi}{2}} = 1 - \frac{1}{3} = \frac{2}{3}$$
$$\int_0^1 2\rho^6\,\mathrm{d}\rho = \left(\frac{2}{7}\rho^7\right)_{\rho=0}^{\rho=1} = \frac{2}{7}$$
$$\int_0^{\frac{\pi}{2}}\cos\phi\sin^3\phi\,\mathrm{d}\phi = \left(\frac{1}{4}\sin^4\phi\right)_{\phi=0}^{\phi=\frac{\pi}{2}} = \frac{1}{4}$$
$$\int_0^1 2\rho^5\,\mathrm{d}\rho = \frac{1}{3}\left(\frac{2}{6}\rho^6\right)_{\rho=0}^{\rho=1} = \frac{1}{3}$$
$$\int_0^{\frac{\pi}{2}}\mathrm{d}\phi = \frac{\pi}{2}$$
$$\int_0^1\rho^3\,\mathrm{d}\rho = \frac{1}{4}$$

So, putting it all together, we arrive at the answer:

$$\Phi = \left(\frac{2}{3}\right)\left(\frac{2}{7}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) - \left(\frac{\pi}{2}\right)\left(\frac{1}{4}\right) = \frac{4}{21} + \frac{1}{12} - \frac{\pi}{8} = \frac{23}{84} - \frac{\pi}{8}$$

Let's repeat the problem but using **Option 1** for the parametrization of the surface: The cartesian parameters will make it easy to find the dot product, but the limits of integration will be a disk, making the evaluation of the integral more challenging:

1. **Parametrize the surface**:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}$$

2. **Find the surface element**. The normal vector is:

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \hat{\mathbf{x}} \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial x}{\partial x} \\ \frac{\partial z}{\partial y} & \frac{\partial x}{\partial y} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} 0 & 2x \\ 1 & 2y \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} 2x & 1 \\ 2y & 0 \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= -2x\,\hat{\mathbf{x}} - 2y\,\hat{\mathbf{y}} + \hat{\mathbf{z}}$$

Therefore:

$$\mathrm{d}\mathbf{S} = \mathbf{N}\,\mathrm{d}x\,\mathrm{d}y = (-2x\,\hat{\mathbf{x}} - 2y\,\hat{\mathbf{y}} + \hat{\mathbf{z}})\,\mathrm{d}x\,\mathrm{d}y$$

Notice that this has given us an orientation of the vector that is opposite to the one we got earlier. This one is pointing toward the inside of the paraboloid. A surface has, of course, two possible definitions of the normal vector. The flux will change sign depending on this arbitrary choice (i.e. calculating the flux from one side into the other, or vice versa).

3. **Find the scalar integrand** (do the dot product $\mathbf{F} \cdot \mathrm{d}\mathbf{S}$)

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (x^2 z, xy^2, z) = \begin{pmatrix} x^2 z \\ xy^2 \\ z \end{pmatrix} \cdot \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix} \mathrm{d}x\,\mathrm{d}y = (-2x^3 z - 2xy^3 + z)\,\mathrm{d}x\,\mathrm{d}y$$

But we must write z in terms of the parameters x, y, so that we substitute $z = x^2 + y^2$.

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = (-2x^5 - 2x^3y^2 - 2xy^3 + x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$$

4. **Perform the double scalar integration**.

Find the limits of integration. In the xy parameter space, the limits of integration are the first quadrant of the unit disk centred in the origin.

$$\Phi = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^1 \left(\int_0^{\sqrt{1-y^2}} (-2x^5 - 2x^3y^2 - 2xy^3 + x^2 + y^2)\,\mathrm{d}x \right) \mathrm{d}y$$

This might be solvable with care and labour, but it seems easier to change the double integration into polar coordinates by using a change of variables, $x = \rho\cos\phi$, $y = \rho\sin\phi$ and don't forget the Jacobian $\mathrm{d}x\,\mathrm{d}y = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi$:

$$\Phi = \int_0^{\pi/2} \left(\int_0^1 (-2\rho^6 \cos^3\phi - 2\rho^5 \cos\phi \sin^3\phi + \rho^3)\,\mathrm{d}\rho \right) \mathrm{d}\phi$$

which is exactly the integration we solved when parametrizing the surface in cylindrical coordinates (but with an opposite sign, due to the arbitrary flip in $\mathbf{N}$).

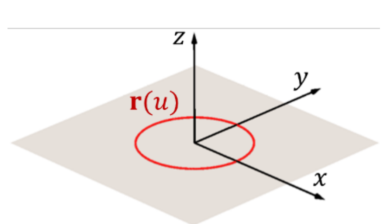
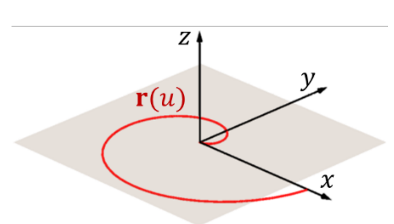
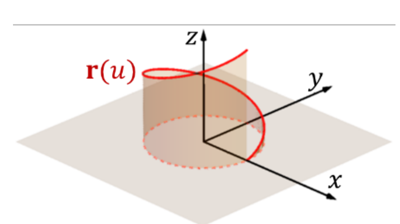
4.4 LINE INTEGRALS

In the same way we can do double integrals in curves surfaces, we can do simple integrals along curved paths: these are called line integrals

A. PARAMETRIZED CURVES

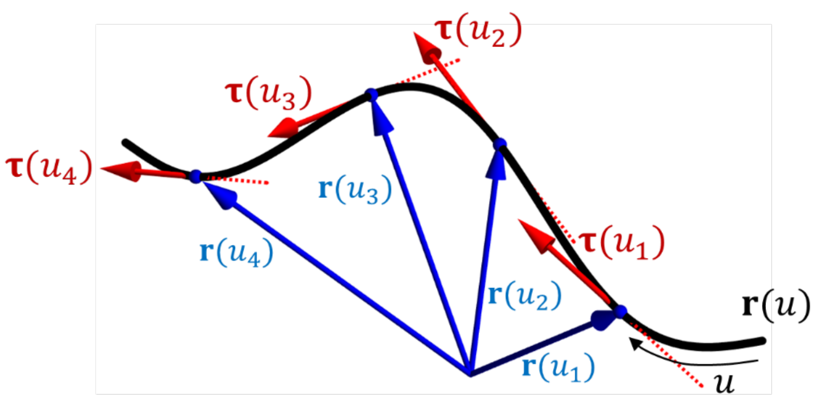
A curve in 3D space can be specified via a function
$\mathbf{r}: (u) \mapsto (x, y, z)$ in a domain $u \in [a, b]$
which maps the real line segment $u \in [a, b]$ into a segment of
curve in 3-D space $\mathbf{r}(u) = (x(u), y(u), z(u))$

Examples of parametrized curves:

$\mathbf{r}(u) = (R\cos u\,, R\sin u\,, 0)$	$\mathbf{r}(u) = (u\cos u\,, u\sin u\,, 0)$	$\mathbf{r}(u) = (R\cos u\,, R\sin u\,, u)$
		

Sometimes curves are given in other forms which need to be parametrized. For instance, the curve $y = x^2$ can be parametrized by taking x as parameter: $\mathbf{r}(u) = (u, u^2, 0)$

TANGENT TO THE CURVE



A vector tangent to the curve $\mathbf{r}(u)$ is given by

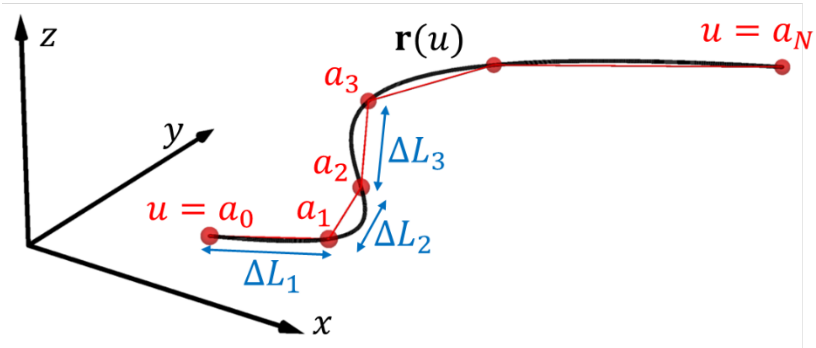
$$\boldsymbol{\tau}(u) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}$$

- The direction of $\boldsymbol{\tau}$ tells us the tangent direction.
- The magnitude of $\boldsymbol{\tau}$ tells us, locally, the distance moved by $\mathbf{r}$ per unit increase in u.

B. LINE INTEGRAL: INTERPRETATION AND APPLICATIONS

Consider a parametrized curve C given by the parametric equation $\mathbf{r}(u)$ with $u \in [a, b]$, and a function $f(\mathbf{r})$ which can be evaluated at the points on the curve $f\big(\mathbf{r}(u)\big) \to f(u)$.

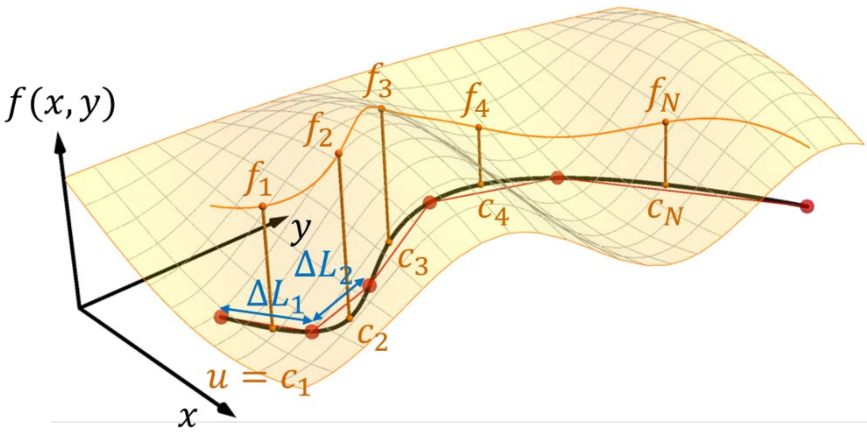
Divide this curve into segments separated by the points corresponding to $u = a_0, a_1, a_2, \dots, a_N$ (with $a_0 = a$ and $a_N = b$). Each segment has a straight-line length $\Delta \mathrm{L}_k = \|\mathbf{r}(a_k) - \mathbf{r}(a_{k-1})\|$. We can evaluate $f(\mathbf{r})$ at points along the curve somewhere inside each segment $f_k = f(\mathbf{r}(c_k))$ where $c_k \in [a_{k-1}, a_k]$.



Then consider the following sum:

$$\sum_{i=1}^{N} f_k \, \Delta L_k$$

We are multiplying the length of the segment, by the function $f(\mathbf{r})$ evaluated at $\mathbf{r}$ in some point of each segment. **This can be visualized when the path $\mathbf{r}(u)$ is 2-D**, because we can use the z axis to represent the function $f(x, y)$ and show how it is sampled at the values $u = c_k$:

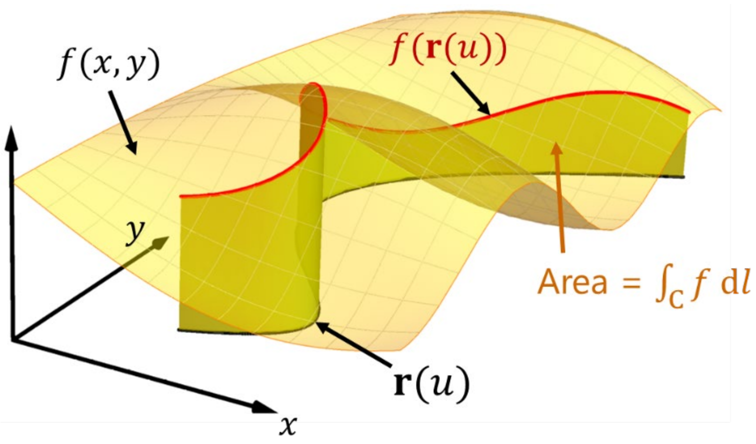


Now take the limit when all the segments $\Delta L_k \to 0$. If the limit exists, it is independent of the subdivisions chosen, and the limit is defined as the **line integral**:

$$\int_C f(\mathbf{r}) \, \mathrm{d}l$$

The line integral lets us add a function over a certain curve. The function can be scalar or a vector.

In two dimensions it is easy to give a geometrical meaning to the line integral. The line integral gives us the area under the graph $f(x,y)$ evaluated along the curve $\mathbf{r}(u) = \big(x(u), y(u)\big)^T$.



In three dimensions it is not possible to visualize it. The function is evaluated at each point in the curve and all values are added along the curve. We can understand it with examples:

Example applications:

The length of a curve C:

$$L = \int_C 1\ \mathrm{d}l$$

The mass of a wire following a curve C parametrised with $\mathbf{r}(u)$ having a linear density $\lambda(u)[\mathrm{kg/m}]$

$$M = \int_C \overbrace{\underbrace{\lambda(u)}_{[\mathrm{kg/m}]}\ \underbrace{\mathrm{d}l}_{[\mathrm{m}]}}^{\mathrm{d}m\ [\mathrm{kg}]}$$

Total force (vector) acting on a wire C (as a function of a "force density" $\mathbf{f}(u)[N/m]$ acting on each differential segment of the wire):

$$\underbrace{\mathbf{F}}_{[\mathrm{N}]} = \int_C \underbrace{\mathbf{f}(u)}_{[\mathrm{N/m}]}\ \underbrace{\mathrm{d}l}_{[\mathrm{m}]}$$

Total electric field created at $\mathbf{r}_\mathrm{a}$ by a linear wire C carrying a density of charge $\lambda(u)$:

$$\underbrace{\mathbf{E}(\mathbf{r}_0)}_{\substack{[N/c\\ =V/m\,]}} = \int_C \underbrace{k_e}_{[\mathrm{N\,m^2\,C^{-2}}]} \underbrace{\frac{\hat{\mathbf{e}}_{r'}}{\|\mathbf{r}'\|^2}}_{[\mathrm{m^{-2}}]} \overbrace{\underbrace{\lambda(u)}_{[\mathrm{C/m}]}\ \underbrace{\mathrm{d}l}_{[\mathrm{m}]}}^{\mathrm{d}q\ [\mathrm{C}]} \quad \text{with} \quad \mathbf{r}' = \mathbf{r}_0 - \mathbf{r}(u)$$

Centre of mass of a wire (curve C) parametrised with $\mathbf{r}(u)$ having a linear density $\lambda(u)[\mathrm{kg/m}]$

$$\mathbf{r}_{\mathrm{CM}} = \frac{\int_C \mathbf{r}(u)\,\lambda(u)\,\mathrm{d}l}{\int_C \lambda(u)\,\mathrm{d}l}$$

Weighted average of quantity $f(u)$ over the curve C parametrised by $\mathbf{r}(u)$, with "weight" $w(u)$

$$f_{av}^{w} = \frac{\int_C f(u)\,w(u)\,\mathrm{d}l}{\int_C w(u)\,\mathrm{d}l}$$

C. LINE INTEGRAL CALCULATION

To perform the integral along the curve C, we just need to map the curve in 3D space into a straight line in one dimensional u-space via a parametrization $\mathbf{r}(u)$. The integral is **independent** on how we choose to do this parametrization. Once the curve is parametrized, we can integrate over the parameter u. To do this, we need to:

- Find the differential line element $\mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}u$ equal to the differential length moved along the curve when the parameter is increased from $u \to u + du$ (Note that this is exactly what the magnitude of the tangent vector $\boldsymbol{\tau}(u) = \mathrm{d}\mathbf{r}/\mathrm{d}u$ is telling us).
- Evaluate the integrand $f(\mathbf{r})$ on the points along the curve, i.e. $f(\mathbf{r}(u)) \to f(u)$

Therefore, the general method is:

$$\int_C f(\mathbf{r})\,\mathrm{d}l = \int_a^b f(\mathbf{r}(u))\,\underbrace{\left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right\|\mathrm{d}u}_{\substack{\mathrm{d}l \\ =\|\boldsymbol{\tau}(u)\|\,\mathrm{d}u}}$$

1) Find a parametrisation of the curve $\mathbf{r}(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}$

2) Find the length differential (also called length element) $\mathrm{d}l$:

i. Remember it or look it up (for simple cases)

ii. Geometrical intuition (consider the length of the segment $\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}$ when $u \to u + du$).

iii. Apply the recipe $\mathrm{d}l = \|\mathrm{d}\mathbf{r}\| = \left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right\|\mathrm{d}u$

3) Evaluate the integrand function $f(\mathbf{r})$ at the locations on the curve by substituting the parametrisation of the curve $\mathbf{r}(u)$ into the integrand $f(\mathbf{r})$ (i.e. obtain $f\big(\mathbf{r}(u)\big) = f(u)$)

4) Calculate the integral in the appropriate limits $u \in [a, b]$

Let's do some examples:

1) Calculate the length of the circle of radius R

Solution: We need to integrate $\mathrm{d}l$ over the circle of radius R:

$$L = \int_{\text{Circle}} 1\,\mathrm{d}l$$

We parametrize the curve using cylindrical coordinates (we can do it in 2D):

$$\mathbf{r}(\phi) = \begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix} \text{ with } \phi \in [0,2\pi]$$

The tangent vector is:

$$\boldsymbol{\tau}(\phi) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi} = \begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix}$$

Therefore, the differential line element is:

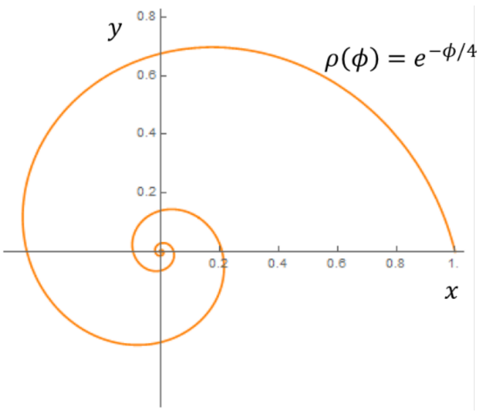
$$\mathrm{d}l = \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi} \right\| \mathrm{d}\phi = \|\boldsymbol{\tau}(\phi)\|\,\mathrm{d}\phi = \sqrt{(R\sin\phi)^2 + (R\cos\phi)^2}\,\mathrm{d}\phi = R\,\mathrm{d}\phi$$

So, we can do the integral in $\phi \in [0,2\pi]$

$$L = \int_{\text{Circle}} 1\,\mathrm{d}l = \int_0^{2\pi} R\,\mathrm{d}\phi = 2\pi R$$

2) Determine the length of the spiral given in polar coordinates as $\rho = e^{-\phi/4}$, with $\phi \in [0,\infty]$.

Solution:



We need to integrate $\mathrm{d}l$ over the spiral:

$$L = \int_{\text{Spiral}} 1\,\mathrm{d}l$$

1. **Parametrize the curve**. We can use polar coordinates with $\rho = e^{-\phi/4}$. Remember $x = \rho\cos\phi\,; y = \rho\sin\phi$.

$$\mathbf{r}(\phi) = \begin{pmatrix} e^{-\phi/4}\cos\phi \\ e^{-\phi/4}\sin\phi \end{pmatrix} \text{ with } \phi \in [0,\infty]$$

2. **Obtain the differential length** $\mathrm{d}l$. The tangent vector is:

$$\boldsymbol{\tau}(\phi) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi} = \begin{pmatrix} -e^{-\frac{\phi}{4}}\sin\phi - \frac{1}{4}e^{-\frac{\phi}{4}}\cos\phi \\ e^{-\frac{\phi}{4}}\cos\phi - \frac{1}{4}e^{-\frac{\phi}{4}}\sin\phi \end{pmatrix} = e^{-\phi/4}\begin{pmatrix} -\sin\phi - \frac{1}{4}\cos\phi \\ \cos\phi - \frac{1}{4}\sin\phi \end{pmatrix}$$

Therefore, the line element is:

$$\mathrm{d}l = \left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi}\right\| \mathrm{d}\phi = \|\boldsymbol{\tau}(\phi)\|\, \mathrm{d}\phi = \mathrm{d}\phi\, e^{-\frac{\phi}{4}} \sqrt{\left(-\sin\phi - \frac{1}{4}\cos\phi\right)^2 + \left(\cos\phi - \frac{1}{4}\sin\phi\right)^2}$$

$$= \mathrm{d}\phi\, e^{-\phi/4} \sqrt{\left(\sin^2\phi + \frac{1}{2}\sin\phi\cos\phi + \frac{1}{16}\cos^2\phi\right) + \left(\cos^2\phi - \frac{1}{2}\sin\phi\cos\phi + \frac{1}{16}\sin^2\phi\right)}$$

$$= \mathrm{d}\phi\, e^{-\phi/4}\sqrt{1 + \frac{1}{16}} = \mathrm{d}\phi\, e^{-\frac{\phi}{4}}\frac{\sqrt{17}}{4}$$

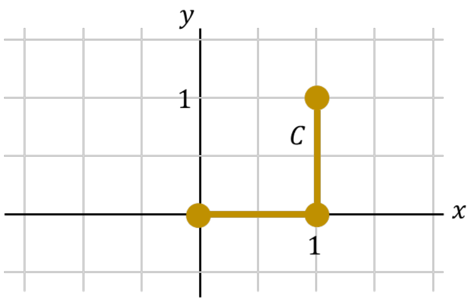
3. **Evaluate integrand at the curve and calculate the integral** in $\phi \in [0, \infty]$

$$L = \int_{\text{Spiral}} 1\,\mathrm{d}l = \int_0^\infty \frac{\sqrt{17}}{4}\, e^{-\phi/4}\,\mathrm{d}\phi = \sqrt{17}\left[-e^{-\phi/4}\right]_{\phi=0}^{\phi=\infty} = \sqrt{17}$$

3) Calculate the line integral $\int_C (x+y)\mathrm{d}l$ where C is the path joining the points $(0,0) \to (1,0)$ in a straight line, followed by the path joining $(1,0) \to (1,1)$ in a straight line.

This integral needs to be done in two parts:

$$\int_C (x+y)\mathrm{d}l = \int_{(0,0)}^{(1,0)} (x+y)\mathrm{d}l + \int_{(1,0)}^{(1,1)} (x+y)\mathrm{d}l$$



1. **Parametrise the two paths**:

$$\mathbf{r}_1(u) = (u, 0)^T$$

$$\mathbf{r}_2(u) = (1, u)^T$$

2. **Find the differential length** $\mathrm{d}l$ for the two paths:

$$\mathrm{d}l_1 = \left\|\frac{\mathrm{d}\mathbf{r}_1}{\mathrm{d}u}\right\| \mathrm{d}u = \left\|\begin{pmatrix}1\\0\end{pmatrix}\right\| \mathrm{d}u = \mathrm{d}u$$

$$\mathrm{d}l_x = \left\|\frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}u}\right\| \mathrm{d}u = \left\|\begin{pmatrix}0\\1\end{pmatrix}\right\| \mathrm{d}u = \mathrm{d}u$$

3. **Evaluate the integrand** $(x+y)$ **on the two curves**:

Path 1, substitute $\mathbf{r}_1(u)$: $\quad (x+y) = u$

Path 2, substitute $\mathbf{r}_2(u)$: $\quad (x+y) = 1+u$

4. **Calculate the integrals**:

$$\int_C (x+y)\mathrm{d}l = \int_{(0,0)}^{(1,0)} (x+y)\mathrm{d}l + \int_{(1,0)}^{(1,1)} (x+y)\mathrm{d}l = \int_0^1 (u)\mathrm{d}u + \int_0^1 (1+u)\mathrm{d}u$$

$$= \left[\frac{u^2}{2}\right]_0^1 + \left[u + \frac{u^2}{2}\right]_0^1 = \frac{1}{2} + \frac{3}{2} = 2$$

D. LINE INTEGRAL OF VECTOR FIELD WITH DOT PRODUCT $\mathbf{F}\cdot\mathbf{dl}$

Surprisingly many times in Physics, equations lead to a line integral along a path L in which the integrand is the **dot product** between a vector field $\mathbf{F}(\mathbf{r})$ and the unit vector tangential to the path $\hat{\mathbf{t}} = \boldsymbol{\tau}/||\boldsymbol{\tau}||$ (with $\boldsymbol{\tau} = \mathrm{d}\mathbf{r}/\mathrm{d}u$):

$$W = \int_L \underbrace{(\mathbf{F}\cdot\hat{\mathbf{t}})}_{\text{dot product}} \; \mathrm{d}l = \int_L \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

$$\mathrm{d}\mathbf{r} \overset{\text{def}}{=} \hat{\mathbf{t}}\,\mathrm{d}l$$

($\hat{\mathbf{t}}$ is the tangent unit vector)

Once the dot product is performed then $(\mathbf{F}\cdot\hat{\mathbf{t}})\mathrm{d}l$ is a regular scalar line integral as studied earlier. Since this integral involves the dot product with the tangential vector, **the direction in which the path is traversed is important** (swapping direction determines the sign of the result).

As an example, the work done by a force is equal to the force multiplied by the displacement $\Delta\mathbf{r}$ **in the direction of the force**. This statement can be written as a dot product when the force is constant along the path: $W = \mathbf{F}\cdot\Delta\mathbf{r}$. However, when the force is changing throughout the path, we can divide it into tiny differential paths and write $\mathrm{d}W = \mathbf{F}(\mathbf{r})\cdot\mathrm{d}\mathbf{r}$ for each, so that the total work done:

$$W = \int_L \mathbf{F}(\mathbf{r})\cdot\mathrm{d}\mathbf{r}$$

CALCULATION

Once a parametrization $\mathbf{r}(u)$ with $u \in [a,b]$ is found, we can find the vector displacement differential $\mathrm{d}\mathbf{r}$ in terms of the parameter u and $\mathrm{d}u$. Two ways to prove this: First, consider the definition $\mathrm{d}\mathbf{r} \overset{\text{def}}{=} \hat{\mathbf{t}}\,\mathrm{d}l$, together with the definition of $\hat{\mathbf{t}} = \boldsymbol{\tau}/||\boldsymbol{\tau}||$ and of the scalar line integral element $\mathrm{d}l = ||\boldsymbol{\tau}||\mathrm{d}u$. Alternatively, calculate **total differential** of the vector function $\mathbf{r}(u)$ using the chain rule $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u$.

$$\mathrm{d}\mathbf{r} = \hat{\mathbf{t}}\,\mathrm{d}l = \frac{\boldsymbol{\tau}}{||\boldsymbol{\tau}||}||\boldsymbol{\tau}||\mathrm{d}u = \boldsymbol{\tau}\,\mathrm{d}u = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u \qquad \leftrightarrow \qquad \mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u$$

Both ways lead to the same result. Therefore, writing this explicitly:

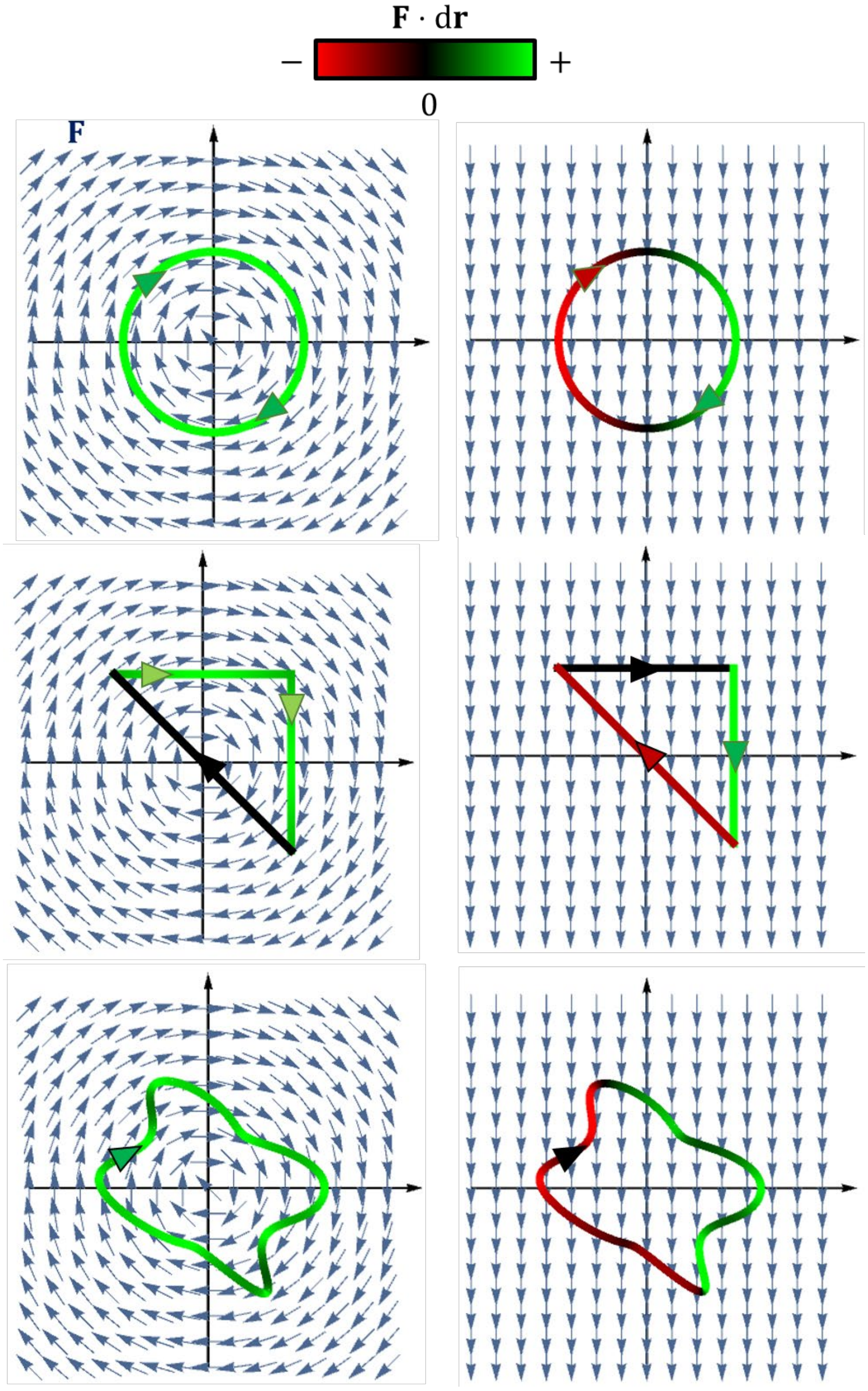
$$W = \int_L \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \left(\mathbf{F}(\mathbf{r}(u))\cdot\frac{\mathrm{d}\mathbf{r}(u)}{\mathrm{d}u}\right)\mathrm{d}u$$

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \begin{pmatrix} \mathrm{d}x/\mathrm{d}u \\ \mathrm{d}y/\mathrm{d}u \\ \mathrm{d}z/\mathrm{d}u \end{pmatrix}$$

1) Parametrise the curve: $\mathbf{r}(u)$ with $u \in [a,b]$.
2) Find the displacement differential $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u$.
3) Evaluate the integrand $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ on the curve. Remember to do the **DOT PRODUCT** and to **SUBSTITUTE** $\mathbf{r}(u)$ into $\mathbf{F}(\mathbf{r})$ so that $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = g(u)\,\mathrm{d}u$ is a scalar integrand involving u only.
4) Calculate the 1D definite integration in $u \in [a,b]$.

VISUAL INTUITION:

The line integral of $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is adding up positive values $(\mathbf{F}\cdot\mathrm{d}\mathbf{r})>0$ when the path moves in the direction of the field, negative values $(\mathbf{F}\cdot\mathrm{d}\mathbf{r})<0$ when the path moves in opposite direction to the field, and does not count (zero value $(\mathbf{F}\cdot\mathrm{d}\mathbf{r})=0$) those regions where the path moves exactly orthogonal to the vector field. See examples:



The direction of the path is important, as flipping the direction $C\to -C$ will change the sign:

$$\int_C \mathbf{F}\cdot \mathrm{dr} = -\int_{-C} \mathbf{F}\cdot \mathrm{dr}$$

4) Consider the 2D vector field $\mathbf{F}(x, y) = (1,1 + x)$. Calculate the integral $\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where L is the circular anticlockwise path with radius R centered at the origin.

Solution:

1) **Parametrise the path**:

$$\mathbf{r}(\phi) = \begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix} \text{ with } \phi \in [0,2\pi]$$

2) **Find the vector path differential** $\mathrm{d}\mathbf{r}$:

$$\boldsymbol{\tau}(\phi) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi} = \begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix}$$

So, the displacement differential can be obtained as:

$$\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi}\mathrm{d}\phi = \begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix}\mathrm{d}\phi$$

3) **Evaluate the integrand $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ along the path** (remember substituting $\mathbf{F}\big(\mathbf{r}(\phi)\big)$ and doing the dot product)

(i) The vector field $\mathbf{F}$ needs to be expressed in terms of the coordinate ϕ by substituting the path $\mathbf{r}(\phi)$ into x and y:

$$\mathbf{F}\big(\mathbf{r}(\phi)\big) = \mathbf{F}(x(\phi), y(\phi)) = \mathbf{F}(R\cos\phi, R\sin\phi) = \begin{pmatrix} 1 \\ 1 + R\cos\phi \end{pmatrix}$$

(ii) The dot product needs to be done:

$$\mathbf{F} \cdot \mathrm{d}\mathbf{r} = \begin{pmatrix} 1 \\ 1 + R\cos\phi \end{pmatrix} \cdot \begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix}\mathrm{d}\phi = (-R\sin\phi + R\cos\phi + R^2\cos^2\phi)\,\mathrm{d}\phi$$

The order of the two steps above can be swapped.

4) **Calculate the integral:**

Everything is written in terms of either constants (R) or parameters (ϕ), so we are ready to integrate in $\phi \in [0,2\pi]$:

$$\begin{aligned}
\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} (-R\sin\phi + R\cos\phi + R^2\cos^2\phi)d\phi \\
&= \int_0^{2\pi} \left(-R\sin\phi + R\cos\phi + R^2\left(\frac{1}{2} + \frac{1}{2}\cos 2\phi\right)\right)d\phi \\
&= \left[R\cos\phi + R\sin\phi + \frac{R^2\phi}{2} + \frac{R^2}{4}\sin 2\phi\right]_{\phi=0}^{\phi=2\pi} \\
&= (R + 0 + R^2\pi + 0) - (R + 0 + 0 + 0) = R^2\pi
\end{aligned}$$

ALTERNATIVE NOTATION (AND POSSIBLE SHORTCUTS)

Start from the usual notation:

$$W = \int_L \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

For the function $\mathbf{F}$, we can write its three components. For the vector differential $\mathrm{d}\mathbf{r}$, consider how the $\mathrm{d}u$'s cancel on each component. We arrive at:

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} F_x(\mathbf{r}) \\ F_y(\mathbf{r}) \\ F_z(\mathbf{r}) \end{pmatrix}; \qquad \mathrm{d}\mathbf{r} = \begin{pmatrix} \mathrm{d}x/\mathrm{d}u \\ \mathrm{d}y/\mathrm{d}u \\ \mathrm{d}z/\mathrm{d}u \end{pmatrix} \mathrm{d}u = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix};$$

Therefore, we can explicitly carry out the dot product of the two vectors $\mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$:

$$\mathbf{F} \cdot \mathrm{d}\mathbf{r} = F_x\, \mathrm{d}x + F_y\, \mathrm{d}y + F_z\, \mathrm{d}z$$

Leading to a very common way of writing vector field line integrals:

$$\boxed{W = \int_L F_x\, \mathrm{d}x + F_y\, \mathrm{d}y + F_z\, \mathrm{d}z = \int_L F_x \mathrm{d}x + \int_L F_y \mathrm{d}y + \int_L F_z \mathrm{d}z}$$

In this notation, **the dot product** (an important part of step 3) **has already been done for you**! So, in step 3, you just have to evaluate the integrand and the differentials along the curve.

The advantage of this last form is that we can use, for instance, x directly as the parameter in the first integral, and then write the x component of $\mathbf{F}$ as a function of that parameter only $F_x(x) = \mathbf{F}\big(\mathbf{r}(x)\big) \cdot \hat{\mathbf{x}}$ so that **we can evaluate the integral directly, with no need to compute tangents nor parametrizing the curve with an external parameter**.

Remember, even though this notation looks different, it is completely equivalent to previous definitions of line integrals, and they always give the same result.

For example, starting from the above, if the path is written as $\mathbf{r}(u) = (x(u), y(u), z(u))$, then it is clear that $\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u = x'(u)\, \mathrm{d}u$, and similarly for the other two, so the integral can be reduced back to where we started:

$$\boxed{W = \int_a^b \left[F_x(u)\, x'(u) + F_y(u)\, y'(u) + F_z(u)\, z'(u)\right]\mathrm{d}u = \int_a^b \left(\mathbf{F}(\mathbf{r}(u)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}(u)\right) \mathrm{d}u}$$

5) Calculate the line integral:

$$W = \int_{\mathrm{L}} x\,\mathrm{d}x - y\,\mathrm{d}y + z\,\mathrm{d}z$$

Along one revolution of the helical line specified parametrically as $x = R\cos\phi\,;y = R\sin\phi\,;z = b\phi$.

Solution:

1. Parametrise the curve, already done in the question. One revolution corresponds to a change of $\phi \in [0,2\pi]$.

2. Find the displacement differentials $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = \mathrm{d}\mathbf{r}$.

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = -R\sin\phi \rightarrow\ \mathrm{d}x = -R\sin\phi\,\mathrm{d}\phi;$$
$$\frac{\mathrm{d}y}{\mathrm{d}\phi} = R\cos\phi \rightarrow\ \mathrm{d}y = R\cos\phi\,\mathrm{d}\phi;$$
$$\frac{\mathrm{d}z}{\mathrm{d}\phi} = b \rightarrow \mathrm{d}z = b\,\mathrm{d}\phi$$

Note that this is equivalent to $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi}\mathrm{d}\phi = \begin{pmatrix} -R\sin\phi \\ R\cos\phi \\ b \end{pmatrix}\mathrm{d}\phi = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}$

3. Evaluate the integrand at the curve:

$$W = \int_{\mathrm{AB}} x\,\mathrm{d}x - y\,\mathrm{d}y + z\,\mathrm{d}z = \int_0^{2\pi}\left(-x\overbrace{R\sin\phi}^{\mathrm{d}x} - y\overbrace{R\cos\phi}^{\mathrm{d}y} + z\,\overbrace{b}^{\mathrm{d}z}\right)\mathrm{d}\phi$$

The dot product has already been done for us. But be careful! We still need to write x, y and z in terms of the parameter, substituting the parametric equation of the line:

$$W = \int_0^{2\pi}\left(-\overbrace{(R\cos\phi)}^{x(\phi)}R\sin\phi - \overbrace{(R\sin\phi)}^{y(\phi)}R\cos\phi + \overbrace{(b\phi)}^{z(\phi)}\,b\right)\mathrm{d}\phi$$
$$= \int_0^{2\pi}(2R^2\cos\phi\sin\phi + b^2\phi)\,\mathrm{d}\phi\ = \left[R^2\sin^2\phi + \frac{b^2}{2}\phi^2\right]_{\phi=0}^{\phi=2\pi} = 0 + 2\pi^2 b^2$$

A. CIRCULATION OF A FIELD

In many cases in physical laws, the path C is a closed path or loop. Then the integral is called the **circulation of the field around the loop**, and the fact that the path is closed is denoted with a circle on the integration symbol:

$$J = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Example:

6) Calculate:

$$J = \oint_L 2xy\,\mathrm{d}x + x^2\mathrm{d}y$$

Along the loop $L = \mathrm{OB} \to \mathrm{BA} \to \mathrm{AO}$ with $\mathrm{O} = (0,0)$, $\mathrm{B} = (1,0)$, $\mathrm{A} = (0,1)$.

Solution: Consider each path separately:

First path OB: Use parameter x in curve $y = 0$. Therefore $\mathrm{d}y = 0$. Remember to substitute x and y.

$$J_{\mathrm{OB}} = \int_0^1 2xy\,\mathrm{d}x = \int_0^1 2(x)(0)\mathrm{d}x = 0$$

Notice how we didn't have to use any extra parameter; we just integrated in x

Second path BA: Use parameter x with $y = 1 - x$. Therefore $\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = -\mathrm{d}x$. **Problem: the parameter** x goes from $x = 1$ to $x = 0$, it **goes down**! That can be confusing, especially because in $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ integrals, the direction is very important. There are two possible solutions:

(i) **Mathematically rigorous solution**: redefine the path using a parameter which increases, e.g. use the straight-line equation $\mathbf{r}(u) = \mathbf{b} + u(\mathbf{a} - \mathbf{b}) = (1 - u, u)$ with $u \in [0,1]$. Now $\mathrm{d}x = -\mathrm{d}u$ and $\mathrm{d}y = \mathrm{d}u$, and we can continue as usual:

$$J_{\mathrm{BA}} = \int_{\mathrm{BA}} 2xy\,\mathrm{d}x + x^2\mathrm{d}y = \int_0^1 2(1-u)(u)(-\mathrm{d}u) + (1-u)^2\,(\mathrm{d}u) = (\dots) = 0$$

(ii) **Easy solution**: Define a path $-L$ which is identical to the path L but reversed in direction. In that case, we can use the original parametrisation, so that path $-L$ is given by $\mathbf{r}(x) = (x, 1-x)$ with $x \in [0,1]$. Then apply the fact that reversing direction flips the sign $\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_{-L} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$:

$$J_{\mathrm{BA}} = \int_L 2xy\,\mathrm{d}x + x^2\mathrm{d}y = -\int_{-L} 2xy\,\mathrm{d}x + x^2\mathrm{d}y = -\int_0^1 2x(1-x)\mathrm{d}x + x^2(-\,\mathrm{d}x)$$
$$= \int_1^0 (2x - 3x^2)\,\mathrm{d}x = [x^2 - x^3]_{x=1}^{x=0} = 0$$

Third path AO: Use parameter y with $x = 0$. Therefore $\mathrm{d}x = 0$. Same problem of y decreasing.

$$J_{\mathrm{AO}} = \int_L 2xy\,\mathrm{d}x + x^2\mathrm{d}y = -\int_{-L} 2xy\,\mathrm{d}x + x^2\mathrm{d}y = -\int_0^1 2(0)(y)(-\mathrm{d}y) + 0^2\,\mathrm{d}y = 0$$

So finally, $J = J_{\mathrm{OB}} + J_{\mathrm{BA}} + J_{\mathrm{AO}} = 0$. This result is not accidental because $\mathbf{F} = (2xy, x^2)$ is the gradient of a scalar field $U = x^2y$ (we will see what this means in the vector analysis chapter)

COMPUTATIONAL EXAMPLE OF CIRCULATION IN PHYSICS

Ampere's law relates the circulation of a static magnetic field in any closed path C with the total electric current which crosses through any surface enclosed by that path, I_{enc}:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 I_{\text{enc}}$$

Consider a current carrying wire at $(x, y) = (0,0)$ carrying a current $I_1\hat{\mathbf{z}}$ in the positive z direction. The magnetic field created by this current is given by (it can be found using Ampere's law):

$$\mathbf{B}_{I_1}(\mathbf{r}) = \frac{\mu_0 I_1}{2\pi\rho}\hat{\mathbf{e}}_\phi$$

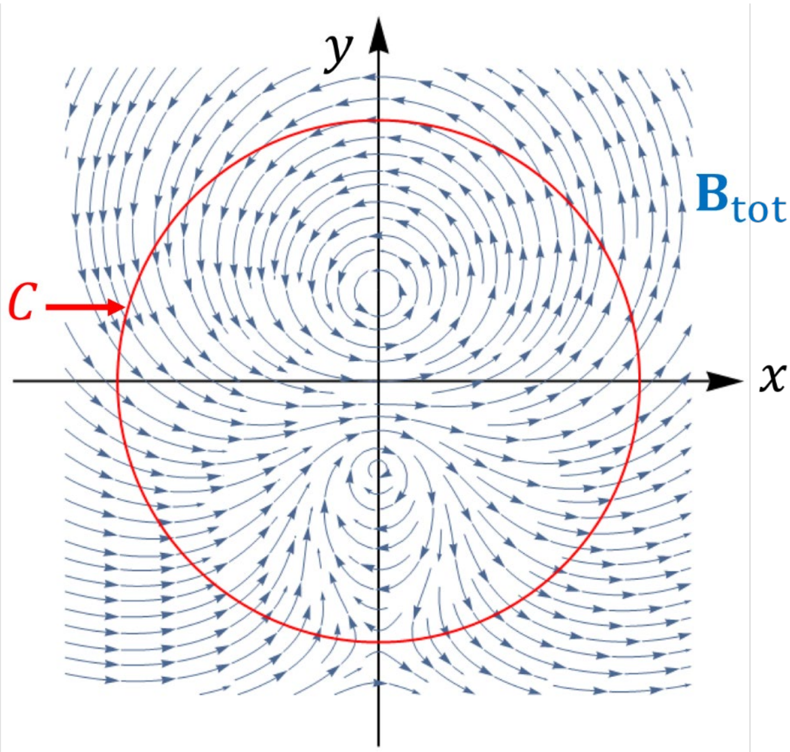
which we can write in cartesian coordinates as: $\mathbf{B}_I(\mathbf{r}) = \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}}\left(\frac{-y\,\hat{\mathbf{x}}+x\,\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}\right) = \frac{\mu_0 I\,(-y\,\hat{\mathbf{x}}+x\,\hat{\mathbf{y}})}{2\pi(x^2+y^2)}$. If the current carrying wire was instead placed at $\mathbf{r}_1 = (x_1, y_1)$, then we can shift our origin, i.e. do the change $x \to x - x_1$, $y \to y - y_1$, so that:

$$\mathbf{B}_{I_1}(\mathbf{r})|_{\text{wire at }(x_1,y_1)} = \frac{\mu_0 I_1\,(-(y-y_1)\,\hat{\mathbf{x}} + (x-x_1)\,\hat{\mathbf{y}})}{2\pi((x-x_1)^2 + (y-y_1)^2)}$$

If we have TWO current carrying wires, wire 1 placed at location $\mathbf{r}_1$ carrying a current $I_1\hat{\mathbf{z}}$, and wire 2 placed at location $\mathbf{r}_2$ carrying a current $I_2\hat{\mathbf{z}}$, the total magnetic field will be the superposition of both:

$$\mathbf{B}_{\text{tot}}(\mathbf{r}) = \mathbf{B}_{I_1}(\mathbf{r})|_{\text{wire at }(x_1,y_1)} + \mathbf{B}_{I_2}(\mathbf{r})|_{\text{wire at }(x_2,y_2)}$$

Computationally it is trivial to add two such functions and simplify the result. For example, this is the magnetic field in the XY plane when a current of $1\hat{\mathbf{z}}$ is placed at $(0,1/2)$ and a current of $-\frac{1}{2}\hat{\mathbf{z}}$ is placed at $(0,-1/2)$: we have $\mathbf{B}_{\text{tot}}(\mathbf{r}) = \mu_0\left(\frac{1-9x^2(-1+y)+y-9y^2-9y^3}{36\pi(x^2+(-\frac{1}{3}+y)^2)(x^2+(\frac{1}{3}+y)^2)}\hat{\mathbf{x}} + \frac{x(\frac{1}{9}+x^2+2y+y^2)}{4\pi(x^2+(-\frac{1}{3}+y)^2)(x^2+(\frac{1}{3}+y)^2)}\hat{\mathbf{y}}\right)$.



Now, let's calculate the circulation of $\mathbf{B}$ around the path C given by the unit circle in the XY plane centred at the origin. For this we follow the steps:

$$\oint_C \mathbf{B}\cdot \mathrm{d}\mathbf{r}$$

1. **Parametrise the path**: $\mathbf{r}(\phi) = (\cos\phi\,, \sin\phi\,, 0)$

2. **Obtain the vector displacement differential**: $\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi}\mathrm{d}\phi = (-\sin\phi\,, \cos\phi\,, 0)$

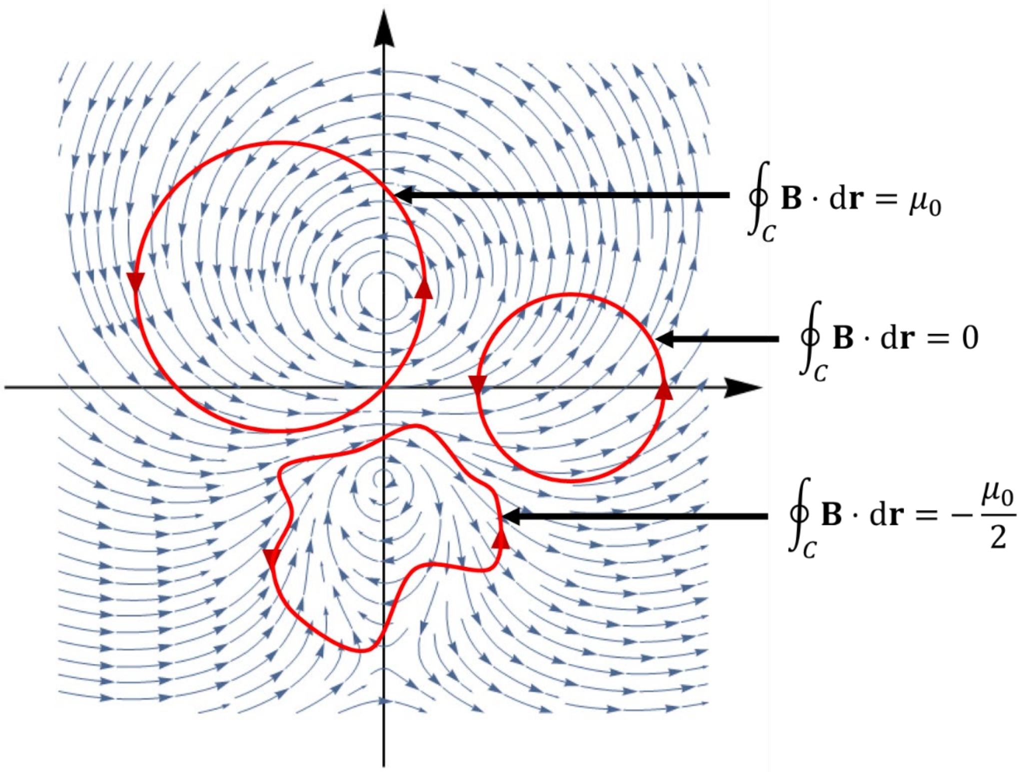
3. **Evaluate the integrand at the curve** and perform the dot product $\mathbf{B}\cdot\mathrm{d}\mathbf{r}$**.** For this we must substitute $x = \cos\phi$ and $y = \sin\phi$ into the expression of $\mathbf{B}$ above, and then do the dot product with $\mathrm{d}\mathbf{r}$. This messy expression can be simplified using the computer, and we arrive at:

$$\mathbf{B}\cdot\mathrm{d}\mathbf{r} = \mu_0 \frac{9(9+\cos^2\phi + 8\sin\phi - \sin^2\phi)}{8\pi(41+9\cos 2\phi)}\mathrm{d}\phi$$

4. **Calculate the integral**. This is a difficult integral, but again, the computer can solve it. The result is:

$$\oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r} = \mu_0\int_0^{2\pi}\frac{9(9+\cos^2\phi + 8\sin\phi - \sin^2\phi)}{8\pi(41+9\cos 2\phi)}\mathrm{d}\phi = (\text{computer}) = \mu_0\frac{1}{2}$$

which is exactly $\mu_0 I_{\text{enc}} = \mu_0(1-\frac{1}{2})$, because our curve enclosed both currents, in accordance to Ampere's law. Performing the integration numerically over different curves with their correct parametrisation, we get always $\mu_0 I_{\text{enc}}$ depending on the total current we enclose:



PROBLEMS

LINE INTEGRALS OF SCALAR FIELDS

7) Calculate $\int_C (x+y)\mathrm{d}l$, where C is the straight-line segment from $(0,0)$ to $(1,1)$.

1. Parametrise the curve. The simplest option is $\mathbf{r}(u) = (u,u)^T$.

2. Find the differential of length: $\mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}u = \left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right\| \mathrm{d}u = \|(1,1)^T\|\mathrm{d}u = \sqrt{2}\,\mathrm{d}u$

3. Evaluate the integrand along the curve: $f(\mathbf{r}(u)) = f(x(u), y(u), z(u))$

$$f(\mathbf{r}(u)) = (x+y)|_{\substack{x=u\\y=u}} = 2u$$

3. Calculate the integral

$$\int_C f\,\mathrm{d}l = \int_0^1 \underbrace{(2u)}_{f(u)} \underbrace{(\sqrt{2}\,\mathrm{d}u)}_{\mathrm{d}l(u)} = 2\sqrt{2}\left[\frac{1}{2}u^2\right]_0^1 = \sqrt{2}$$

8) Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1,1,1)$.

1. Parametrise the curve. The simplest option is: $\mathbf{r}(u) = (u,u,u)^T$ with $u \in [0,1]$

2. Find the differential of length: $\mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}u = \left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right\| \mathrm{d}u = \|(1,1,1)^T\|\mathrm{d}u = \sqrt{3}\,\mathrm{d}u$

3. Evaluate the integrand along the curve: $f(\mathbf{r}(u)) = f(x(u), y(u), z(u))$

$$f(\mathbf{r}(u)) = (x - 3y^2 + z)|_{\substack{x=u\\y=u\\z=u}} = u - 3u^2 + u = 2u - 3u^2.$$

4. Calculate the integral

$$\int_C f\,\mathrm{d}l = \int_0^1 \underbrace{(2u - 3u^2)}_{f(u)} \underbrace{(\sqrt{3}\,\mathrm{d}u)}_{\mathrm{d}l(u)} = \sqrt{3}[u^2 - u^3]_0^1 = 0$$

9) Evaluate $\int_C (x+y+z)\mathrm{d}l$, where C is the straight-line segment from $(0,1,0)$ to $(1,0,0)$.

Solution:

1. Parametrise the curve. The simplest option is: $\mathbf{r}(u) = (u, 1-u, 0)^T$ with $u \in [0,1]$

2. Find the differential of length: $\mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}u = \left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right\| \mathrm{d}u = \|(1,-1,0)^T\|\mathrm{d}u = \sqrt{2}\,\mathrm{d}u$

3. Evaluate the integrand along the curve: $f(\mathbf{r}(u)) = f(x(u), y(u), z(u))$

$$f(\mathbf{r}(u)) = (x+y+z)|_{\substack{x=u\\y=1-u\\z=0}} = u + 1 - u = 1$$

4. Calculate the integral

$$\int_C f\,\mathrm{d}l = \int_0^1 \underbrace{(1)}_{f(u)} \underbrace{(\sqrt{2}\,\mathrm{d}u)}_{\mathrm{d}l(u)} = \sqrt{2}[u]_0^1 = \sqrt{2}$$

10) Calculate the total mass of a circular loop (given by the unit circle in the XY plane centred at the origin) given that the circle is denser for higher x values according to the linear mass density $\lambda = 2 + x$.

To calculate the total mass, we need to integrate the density along the curve:

$$M = \int_C \lambda \, \mathrm{d}l$$

1. Parametrise the curve: use polar coordinates with $\rho = 1$, so we have $\mathbf{r}(\phi) = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$.

2. Find the differential length. We are using polar coordinates, so by geometrical intuition:

$$\mathrm{d}l = \rho \, \mathrm{d}\phi|_{\rho=1} = \mathrm{d}\phi$$

3. Evaluate the integrand at the curve:

$$\lambda(\mathbf{r}) = 2 + x \;\rightarrow\; \lambda\big(\mathbf{r}(\phi)\big) = \lambda(x = \cos\phi) = 2 + \cos\phi$$

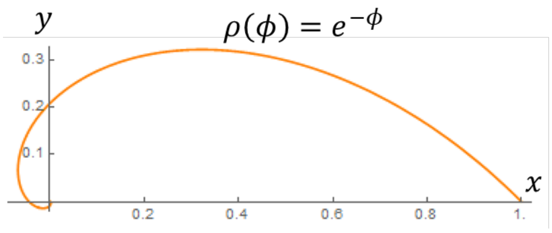
4. Calculate the integral:

$$M = \int_C \lambda \, \mathrm{d}l = \int_0^{2\pi} (2 + \cos\phi) \, \mathrm{d}\phi = [2\phi + \sin\phi]_0^{2\pi} = 4\pi$$

The same as if it had been a constant density of 2. That is because the excess mass in one side is exactly compensated by the lack of mass in the other, due to the linear dependence on x.

11) Determine the length of the spiral given in polar coordinates as $\rho = e^{-\phi}$, with $\phi \in [0, \infty]$.

Solution:



We need to integrate $\mathrm{d}l$ over the spiral:

$$L = \int_{\text{Spiral}} 1 \, \mathrm{d}l$$

1. **Parametrize the curve**. We can use polar coordinates with $\rho = e^{-\phi}$. Remember $x = \rho\cos\phi\,; y = \rho\sin\phi$.

$$\mathbf{r}(\phi) = \begin{pmatrix} e^{-\phi}\cos\phi \\ e^{-\phi}\sin\phi \end{pmatrix} \text{ with } \phi \in [0, \infty]$$

2. **Find the differential line element** $\mathrm{d}l$. The tangent vector is:

$$\boldsymbol{\tau}(\phi) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\phi} = \begin{pmatrix} -e^{-\phi}\sin\phi - e^{-\phi}\cos\phi \\ e^{-\phi}\cos\phi - e^{-\phi}\sin\phi \end{pmatrix} = e^{-\phi}\begin{pmatrix} -\sin\phi - \cos\phi \\ \cos\phi - \sin\phi \end{pmatrix}$$

Therefore, the line element is:

$$\begin{aligned} \mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}\phi &= \mathrm{d}\phi\, e^{-\phi}\sqrt{(-\sin\phi - \cos\phi)^2 + (\cos\phi - \sin\phi)^2} \\ &= \mathrm{d}\phi\, e^{-\phi}\sqrt{(\sin^2\phi + 2\sin\phi\cos\phi + \cos^2\phi) + (\cos^2\phi - 2\sin\phi\cos\phi + \sin^2\phi)} \\ &= \mathrm{d}\phi\, e^{-\phi}\sqrt{2} \end{aligned}$$

3. **Evaluate the integrand at the curve and calculate the integral:**

So we can do the integral in $\phi \in [0, \infty]$

$$L = \int_{\text{Spiral}} 1\, \mathrm{d}l = \int_0^\infty \sqrt{2}\, e^{-\phi}\, \mathrm{d}\phi = \sqrt{2}\left[-e^{-\phi}\right]_{\phi=0}^{\phi=\infty} = \sqrt{2}$$

12) Calculate the line integral:

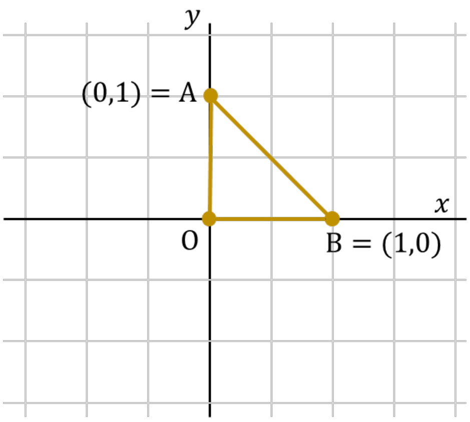
$$I = \oint_L (x + 2y)\mathrm{d}l$$

Over the path $L = \mathrm{OB} \to \mathrm{BA} \to \mathrm{AO}$ with $\mathrm{O} = (0,0), \mathrm{B} = (1,0), \mathrm{A} = (0,1)$.

Solution:

Note: the circle on the integral symbol $\oint_L (x + 2y)\mathrm{d}l$ is a way of indicating that **the path is closed**.

We cannot parametrize this curve as a single function $\mathbf{r}(u)$. Instead we can split the integral into its three sections or contributions.



$$I = \oint_L (x + 2y)\mathrm{d}l = \int_{\mathrm{OB}} (x + 2y)\mathrm{d}l + \int_{\mathrm{BA}} (x + 2y)\mathrm{d}l + \int_{\mathrm{AO}} (x + 2y)\mathrm{d}l$$

First path OB: Let's use $x \in [0,1]$ as parameter.

$$\mathbf{r}(x) = \begin{pmatrix} x \\ 0 \end{pmatrix} \to \boldsymbol{\tau}(x) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}x = \mathrm{d}x$$

$$\int_{\mathrm{OB}} (x + 2y)\big|_{\substack{x=x \\ y=0}} \mathrm{d}l = \int_0^1 x\, \mathrm{d}x = \frac{1}{2}$$

Second path BA: Let's use $y \in [0,1]$ as parameter.

$$\mathbf{r}(y) = \begin{pmatrix} 1-y \\ y \end{pmatrix} \to \boldsymbol{\tau}(y) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}y} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \to \mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}x = \sqrt{2}\,\mathrm{d}y$$

$$\int_{\mathrm{OB}} (x+2y)|_{\substack{x=1-y \\ y=y}} \sqrt{2}\,\mathrm{d}l = \int_0^1 (1-y+2y)\sqrt{2}\,\mathrm{d}x = \sqrt{2}\left[y - \frac{y^2}{2} + y^2\right]_{y=0}^{y=1} = \sqrt{2}\left(\frac{3}{2}\right)$$

Third path AO: It seems we should use $\mathbf{r}(y) = \begin{pmatrix} 0 \\ y \end{pmatrix}$ with $y \in [1,0]$. **But warning!! We have a parameter running backwards!** What do we do with this? There are two solutions:

(i) **Mathematically rigorous solution**: Let's choose a parameter which really grows with the integration path. Let's define $y = 1 - u$ and use $u \in [0,1]$ as the parameter:

$$\mathbf{r}(u) = \begin{pmatrix} 0 \\ 1-u \end{pmatrix} \to \boldsymbol{\tau}(u) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \to \mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}u = \mathrm{d}u$$

$$\int_{\mathrm{OB}} (x+2y)|_{\substack{x=0 \\ y=1-u}} \mathrm{d}l = \int_0^1 (2-2u)\,\mathrm{d}x = (2u-u^2)_{u=0}^{u=1} = 1$$

(ii) **Fast (but potentially more confusing) solution:** The definition of line integrals of the type $\int_C f(u)\mathrm{d}u$ does not care about the direction in which we traverse the path, so if we define a path $-C$ which is exactly equal to C but traversed in the opposite direction, the result of the integral must be unchanged $\int_C f(u)\mathrm{d}u = \int_{-C} f(u)\mathrm{d}u$. So, we can use $-C$ with the increasing parameter $y \in [0,1]$ (But warning: this is exactly the opposite of what happens in integrals of the type $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ seen later, which do change sign. If in doubt, the **mathematically rigorous solution** always works.)

$$\mathbf{r}(y) = \begin{pmatrix} 0 \\ y \end{pmatrix} \to \boldsymbol{\tau}(x) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}y = \mathrm{d}y$$

$$\int_{\mathrm{OB}} (x+2y)|_{\substack{x=0 \\ y=y}} \mathrm{d}l = + \int_{\mathrm{BO}} (x+2y)|_{\substack{x=0 \\ y=y}} \mathrm{d}l = \int_0^1 2y\,\mathrm{d}y = (y^2)^{y=1} = 1$$

So finally, adding up the three paths:

$$I = \oint_L (x+2y)\mathrm{d}l = \int_{\mathrm{OB}} (x+2y)\mathrm{d}l + \int_{\mathrm{BA}} (x+2y)\mathrm{d}l + \int_{\mathrm{AO}} (x+2y)\mathrm{d}l = \frac{1}{2} + \frac{3\sqrt{2}}{2} + 1$$
$$= \frac{3(1+\sqrt{2})}{2}$$

Note: There is a shortcut when using x as the parameter. We could have directly applied the fact $\mathrm{d}l = \left\|\left(\frac{\mathrm{d}x}{\mathrm{d}x}, \frac{\mathrm{d}y}{\mathrm{d}x}\right)\right\| \mathrm{d}x$, which can be immediately written as:

$$\mathrm{d}l = \mathrm{d}x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \mathrm{d}x\sqrt{1 + (y'(x))^2} = 1\,\mathrm{d}x \text{ for OB where } y(x) = 0$$

and similarly, when using y as parameter:

$$\mathrm{d}l = \mathrm{d}y\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} = \mathrm{d}y\sqrt{1 + (x'(y))^2} = \sqrt{2}\,\mathrm{d}y \text{ for BA where } x(y) = -y$$

which some books give as an equation when integrating directly using x or y; but I think it's better to follow always the same logical steps above and the time saved using this shortcut is not that much.

LINE INTEGRALS OF VECTORS

13) Find the centre of mass of a semi-circular metal arch, given by the equation $y^2 + z^2 = 1$ with $z \geq 0$, knowing that it is denser at the bottom than at the top, following the mass density equation $\lambda(z) = 2 - z$

The centre of mass is calculated as the weighted average of the position vector $\mathbf{r}$ weighted by the density λ, as follows:

$$\mathbf{r}_{\mathrm{CM}} = \frac{\int_C \mathbf{r}\,\lambda\,\mathrm{d}l}{\int_C \lambda\,\mathrm{d}l}$$

The numerator is the **line integral of a vector**, whose result is a vector, which can be solved by splitting it into its components:

$$\int_C \mathbf{r}\,\lambda\,\mathrm{d}l = \int_C (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})\,\lambda\,\mathrm{d}l = \hat{\mathbf{x}}\int_C x\,\lambda\,\mathrm{d}l + \hat{\mathbf{y}}\int_C y\,\lambda\,\mathrm{d}l + \hat{\mathbf{x}}\int_C z\,\lambda\,\mathrm{d}l$$

So, in total, we need to calculate 4 different line integrals. Perform the steps for each one (they share steps 1 and 2):

1. **Parametrise the curve**. Let's use an angle from 0 to π as the parameter:

$$\mathbf{r}(\alpha) = \begin{pmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}$$

2. **Find the differential length**:

$$\mathrm{d}l = \|\boldsymbol{\tau}\|\mathrm{d}\alpha = \left\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\alpha}\right\|\mathrm{d}\alpha = \left\|\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix}\right\|\mathrm{d}\alpha = \sqrt{(-\sin\alpha)^2 + (\cos\alpha)^2}\,\mathrm{d}\alpha = \mathrm{d}\alpha$$

3. **Evaluate the integrand at the curve**. Simply substitute $x = 0, y = \cos\alpha$ and $z = \sin\alpha$.

$$\begin{aligned} \lambda\,\mathrm{d}l &= (2-z)\mathrm{d}l = (2-\sin\alpha)\mathrm{d}\alpha \\ x\,\lambda\,\mathrm{d}l &= x\,(2-z)\,\mathrm{d}l = 0\,\mathrm{d}\alpha \\ y\,\lambda\,\mathrm{d}l &= y\,(2-z)\,\mathrm{d}l = \cos\alpha\,(2-\sin\alpha)\,\mathrm{d}\alpha \\ z\,\lambda\,\mathrm{d}l &= z\,(2-z)\,\mathrm{d}l = \sin\alpha\,(2-\sin\alpha)\,\mathrm{d}\alpha \end{aligned}$$

4. **Calculate the four integrals**:

$$\int_C \lambda\,\mathrm{d}l = \int_0^\pi (2-\sin\alpha)\mathrm{d}\alpha = [2\alpha + \cos\alpha]_0^\pi = (2\pi - 1) - (0+1) = 2\pi - 2$$

$$\int_C x\,\lambda\,\mathrm{d}l = \int_C 0\,\mathrm{d}\alpha = 0$$

$$\begin{aligned} \int_C y\,\lambda\,\mathrm{d}l &= \int_0^\pi \cos\alpha\,(2-\sin\alpha)\,\mathrm{d}\alpha = \int_0^\pi (2\cos\alpha - \sin\alpha\cos\alpha)\,\mathrm{d}\alpha = \left[2\sin\alpha + \frac{1}{2}\cos^2\alpha\right]_0^\pi \\ &= \left(0 + \frac{1}{2}(-1)^2\right) - \left(0 + \frac{1}{2}(1)^2\right) = 0 \end{aligned}$$

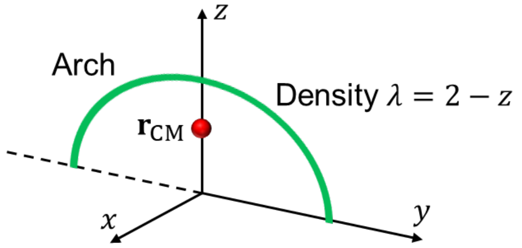
$$\int_C z\,\lambda\,\mathrm{d}l = \int_0^{\pi} \sin\alpha\,(2-\sin\alpha)\,\mathrm{d}\alpha = \int_0^{\pi} (2\sin\alpha - \sin^2\alpha)\,\mathrm{d}\alpha$$

$$= \int_0^{\pi}\left(2\sin\alpha - \left(\frac{1}{2}-\frac{1}{2}\cos 2\alpha\right)\right)\mathrm{d}\alpha = \left[-2\cos\alpha - \frac{\alpha}{2} + \frac{1}{4}\sin 2\alpha\right]_0^{\pi}$$

$$= \left(2-\frac{\pi}{2}+0\right) - (-2-0+0) = 4-\frac{\pi}{2}$$

Hence, we now know all the required integrals:

$$\mathbf{r}_{\mathrm{CM}} = \frac{\int_C \mathbf{r}\,\lambda\,\mathrm{d}l}{\int_C \lambda\,\mathrm{d}l} = \frac{\hat{\mathbf{x}}\int_C x\,\lambda\,\mathrm{d}l + \hat{\mathbf{y}}\int_C y\,\lambda\,\mathrm{d}l + \hat{\mathbf{x}}\int_C z\,\lambda\,\mathrm{d}l}{\int_C \lambda\,\mathrm{d}l} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + \frac{4-\pi/2}{2\pi-2}\hat{\mathbf{z}} = \frac{8-\pi}{4\pi-4}\hat{\mathbf{z}}$$



LINE INTEGRAL OF VECTOR FIELD (WITH DOT PRODUCT $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$)

14) Find the work done by the force $\mathbf{F} = (y-x^2)\hat{\mathbf{x}} + (z-y^2)\hat{\mathbf{y}} + (x-z^2)\hat{\mathbf{z}}$ over the curve $\mathbf{r}(t) = (t, t^2, t^3)$ with $t \in [0,1]$.

1. **Parametrise the curve**. This is already done by the question:

$$\mathbf{r}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \text{ with } t \in [0,1]$$

2. **Find the vector differential of path $\mathrm{d}\mathbf{r}$**:

$$\mathrm{d}\mathbf{r} = \boldsymbol{\tau}\,\mathrm{d}t = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\mathrm{d}t = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}\mathrm{d}t$$

3. **Evaluate the integrand $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$** along the curve, remember to do the dot product:

$$\mathbf{F}(\mathbf{r}(u))\cdot\mathrm{d}\mathbf{r} = \mathbf{F}(x(u),y(u),z(u))\cdot\mathrm{d}\mathbf{r} = \begin{pmatrix} y-x^2 \\ z-y^2 \\ x-z^2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}\mathrm{d}t$$

We can do the dot product first, and then substitute $x=t$, $y=t^2$ and $z=t^3$, or the other way around, both give the same result

$$\mathbf{F}\cdot\mathrm{d}\mathbf{r} = [(t^2-t^2) + (t^3-t^4)(2t) + (t-t^6)(3t^2)]\mathrm{d}t = (2t^4 - 2t^5 + 3t^3 - 3t^8)\mathrm{d}t$$

4. **Calculate the integral**:

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\mathrm{d}t = \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right]_0^1 = \frac{29}{60}$$

The units would be Joules, assuming the force is in Newtons and the distance in meters.

15) Find $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} = x\,\hat{\mathbf{x}} + z\,\hat{\mathbf{y}} + y\,\hat{\mathbf{z}}$ and C is the helical path:

$$\mathbf{r}(t) = \cos t\ \hat{\mathbf{x}} + \sin t\,\hat{\mathbf{y}} + t\,\hat{\mathbf{z}} \text{ with } t \in [0, \pi/2]$$

Note that the integrand is the dot product. It is not a vector. Hence the result is a scalar, not a vector.

1. **Parametrise the curve**. This is already done by the question:

$$\mathbf{r}(t) = \begin{pmatrix}\cos t\\ \sin t\\ t\end{pmatrix} \text{ with } t \in [0, \pi/2]$$

2. **Find the vector differential of path** $\mathrm{d}\mathbf{r}$:

$$\mathrm{d}\mathbf{r} = \boldsymbol{\tau}\,\mathrm{d}t = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\mathrm{d}t = \begin{pmatrix}-\sin t\\ \cos t\\ 1\end{pmatrix}\mathrm{d}t$$

3. **Evaluate the integrand $\mathbf{F}\cdot \mathrm{d}\mathbf{r}$** along the curve, remember to do the dot product:

$$\mathbf{F}\big(\mathbf{r}(u)\big)\cdot \mathrm{d}\mathbf{r} = \mathbf{F}\big(x(u), y(u), z(u)\big)\cdot \mathrm{d}\mathbf{r} = \begin{pmatrix}x\\ z\\ y\end{pmatrix}\cdot\begin{pmatrix}-\sin t\\ \cos t\\ 1\end{pmatrix}\mathrm{d}t$$

We can do the dot product first, and then substitute $x = \cos t$, $y = \sin t$ and $z = t$, or the other way around, both give the same result

$$\mathbf{F}\cdot \mathrm{d}\mathbf{r} = (-x\sin t + z\cos t + y)\mathrm{d}t = (-\cos t\sin t + t\cos t + \sin t)\mathrm{d}t$$

4. **Calculate the integral**:

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^{\pi/2} (-\cos t \sin t + t\cos t + \sin t)\mathrm{d}t$$

Remember the chain rule: $\frac{\mathrm{d}}{\mathrm{d}t}\cos^2 t = -2\sin t\cos t$

Remember the product rule: $\frac{\mathrm{d}}{\mathrm{d}t}(t\sin t) = \frac{\mathrm{d}}{\mathrm{d}t}(t)\sin t + t\,\frac{\mathrm{d}}{\mathrm{d}t}(\sin t) = \sin t + t\cos t$

So the integral can be solved directly once we recognise those patterns.

$$\int_0^{\pi/2} (-\cos t\sin t + t\cos t + \sin t)\mathrm{d}t = \left[\frac{1}{2}\cos^2 t + t\sin t\right]_0^{\pi/2} = (0 + \pi/2) - \left(\frac{1}{2}\right) = \frac{\pi - 1}{2}$$

16) Calculate $\int_C y\ \mathrm{d}x + x^2\mathrm{d}y$ along the following curves:

(i) the parabolic curve given by $y = 4x - x^2$ running from point $(4,0)$ to $(1,3)$

(ii) the straight-line segment running from point $(4,0)$ to $(1,3)$

<u>(i) Parabolic curve path</u>. Notice that this curve is going from right to left.

1. **Parametrise the curve**:

We can try to use x as a parameter.

$$\mathbf{r}(x) = \begin{pmatrix}x\\ 4x - x^2\end{pmatrix} \text{ with } x \in [4,1]$$ **(warning! Parameter going backwards*)**

***How to solve it:** The problem here is that the curve goes from $x = 4$ to $x = 1$, so we should be very careful with using parameters going backwards. There are two options to solve this.

Option 1: Mathematically rigorous way. Define a new parameter u which grows from 1 to 4, so we can define $x = 5 - u$ such that $u = 1$ corresponds to $x = 4$ and viceversa.

$$\mathbf{r}(u) = \begin{pmatrix} 5-u \\ 4(5-u)-(5-u)^2 \end{pmatrix} = \begin{pmatrix} 5-u \\ -u^2+6u-5 \end{pmatrix}$$

Option 2: Easy way. We define a curve $-C$ which corresponds to curve C but going in reverse. Therefore, the sign of the integral will need to be changed later $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\int_{-C} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$.

$$\text{Curve } -C\text{: } \mathbf{r}(x) = \begin{pmatrix} x \\ 4x - x^2 \end{pmatrix} \text{ with } x \in [1,4]$$

Let's use option 2.

2. **Find the differentials of length** $\mathrm{d}x$ and $\mathrm{d}y$ in terms of the parameter x and its differential $\mathrm{d}x$:

$$\mathrm{d}x = \mathrm{d}x$$

$$y = 4x - x^2 \rightarrow \mathrm{d}y = (4-2x)\mathrm{d}x$$

Note that this is equivalent to finding the vector differential of path in the usual way:

$$\mathrm{d}\mathbf{r} = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x}\mathrm{d}x = \begin{pmatrix} 1 \\ 4-2x \end{pmatrix}\mathrm{d}x$$

3. **Evaluate the integrand $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$** along the curve, the dot product is already done.

$$y\,\mathrm{d}x + x^2\mathrm{d}y = \begin{Bmatrix} x = x \\ y = 4x - x^2 \\ \mathrm{d}y = (4-2x)\mathrm{d}x \end{Bmatrix} = (4x - x^2)\mathrm{d}x + x^2(4-2x)\mathrm{d}x = (-2x^3 + 3x^2 + 4x)\mathrm{d}x$$

4. **Calculate the integral**: (**remember** that we are using the path $-C$ so we need to invert the sign)

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_{-C} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_1^4 (-2x^3 + 3x^2 + 4x)\mathrm{d}x = -\left[-\frac{1}{2}x^4 + x^3 + 2x^2\right]_1^4 = \frac{69}{2}$$

<u>**(ii) Straight line path**</u> from point (4,0) to (1,3)

1. **Parametrise the curve**. We need the equation of a straight line going from $\mathbf{r}_1$ to $\mathbf{r}_2$:

$$\mathbf{r}(u) = \mathbf{r}_1 + u(\mathbf{r}_2 - \mathbf{r}_1) = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + u\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4-3u \\ 3u \end{pmatrix} \text{ with } u \in [0,1]$$

2. **Find the differentials** $\mathrm{d}x$ and $\mathrm{d}y$ as a function of the parameter u and $\mathrm{d}u$:

$$x = 4 - 3u \rightarrow \mathrm{d}x = -3\,\mathrm{d}u$$

$$y = 3u \rightarrow \mathrm{d}y = 3\,\mathrm{d}u$$

Note that this is equivalent to finding the vector differential of path in the usual way:

$$\mathrm{d}\mathbf{r} = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u = \begin{pmatrix} -3 \\ 3 \end{pmatrix}\mathrm{d}u$$

3. **Evaluate the integrand $\mathbf{F}\cdot\mathbf{dr}$** along the curve, the dot product is already done.

$$y\,\mathrm{d}x + x^2\mathrm{d}y = \begin{Bmatrix} x = 4-3u \\ y = 3u \\ \mathrm{d}x = -3\,\mathrm{d}u \\ \mathrm{d}y = 3\,\mathrm{d}u \end{Bmatrix} = (3u)(-3\,\mathrm{d}u) + (4-3u)^2(3\,\mathrm{d}u)$$

$$= (-9u + 48 - 72u + 27u^2)\mathrm{d}u$$
$$= (48 - 81u + 27u^2)\mathrm{d}u$$

4. **Calculate the integral**:

$$\int_C y\,\mathrm{d}x + x^2\mathrm{d}y = \int_0^1 (48 - 81u + 27u^2)\mathrm{d}u = \left[48u - \frac{81}{2}u^2 + \frac{27}{3}u^3\right]_0^1 = \frac{33}{2}$$

17) Evaluate $\int_C xy\,\mathrm{d}x + (x+y)\mathrm{d}y$ along the curve $y = x^2$ from $(-1,1)$ to $(1,1)$

Since we have the integrand in the form $F_x\,\mathrm{d}x + F_y\,\mathrm{d}y$, the best parametrisation is to use either x or y as a parameter. Let's do both:

Option A: Use x as parameter.

1. **Parametrise the curve**: $\mathbf{r}(x) = (x, x^2)$ with $x \in [-1,1]$

2. **Find the differentials** $y = x^2 \rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ hence $\mathrm{d}y = 2x\,\mathrm{d}x$

3. **Evaluate the integrand at the curve** (i.e. substitute $y = x^2$ and $\mathrm{d}y = 2x\,\mathrm{d}x$):

$$xy\,\mathrm{d}x + (x+y)\mathrm{d}y = x^3\,\mathrm{d}x + (x + x^2)\,2x\,\mathrm{d}x = (3x^3 + 2x^2)\,\mathrm{d}x$$

4. **Calculate the integral**:

$$\int_C xy\,\mathrm{d}x + (x+y)\mathrm{d}y = \int_{-1}^1 (3x^3 + 2x^2)\,\mathrm{d}x = \left[\frac{3}{4}x^4 + \frac{2}{3}x^3\right]_{-1}^1 = \left(\frac{3}{4} + \frac{2}{3}\right) - \left(\frac{3}{4} - \frac{2}{3}\right) = \frac{4}{3}$$

Option B: Use y as parameter. This gets messy (Option A is the wiser choice)

1. **Parametrise the curve**:

Not being careful we could write $\mathbf{r}(y) = (\sqrt{y}, y)$ with $y \in [1,1]$, but notice that something is wrong with the limits. The problem is that to parametrise the parabolic curve $y = x^2$ using y as a parameter is problematic, because each value of y corresponds to two values of x. Hence, we need to divide the path into two parts, to have 1-to-1 correspondence, and integrate separately:

$\mathbf{r}_1(y) = (-\sqrt{y}, y)$ with $y \in [1,0]$ **(warning! Parameter going backwards – see last problem)**

$\mathbf{r}_2(y) = (+\sqrt{y}, y)$ with $y \in [0,1]$

2. **Find the differentials**:

$$\text{For path } \mathbf{r}_1(u)\text{: } x = -\sqrt{y} \rightarrow \mathrm{d}x = -\frac{1}{2\sqrt{1-u}}\mathrm{d}y$$

$$\text{For path } \mathbf{r}_2(y)\text{: } x = \sqrt{y} \rightarrow \mathrm{d}x = \frac{1}{2\sqrt{y}}\mathrm{d}y$$

3. **Evaluate the integrand at the curve**:

For path $\mathbf{r}_1(u)$: $xy\,\mathrm{d}x + (x+y)\mathrm{d}y = -\sqrt{y}\,y\,\frac{\mathrm{d}u}{-2\sqrt{y}} + \left(-\sqrt{y}+y\right)\mathrm{d}y = \left(\frac{3}{2}y - \sqrt{y}\right)\mathrm{d}y$

For path $\mathbf{r}_2(u)$: $xy\,\mathrm{d}x + (x+y)\mathrm{d}y = \sqrt{y}\,y\,\frac{1}{2\sqrt{y}}\mathrm{d}y + \left(\sqrt{y}+y\right)\mathrm{d}y = \left(\frac{3}{2}y + \sqrt{y}\right)\mathrm{d}y$

4. **Calculate the integral**:

We had a parameter running backwards, therefore we can define curve $-C_1$ with $y \in [0,1]$ as being in opposite direction to C_1, and apply the property: $\int_{C_1} = -\int_{-C_1}$, so we can revert the path, to have y growing, but knowing that we needed to invert the sign of the integral.

$$\int_C xy\,\mathrm{d}x + (x+y)\mathrm{d}y = \int_{C_1} + \int_{C_2} = -\int_{-C_1} + \int_{C_1} = -\int_0^1 \left(\frac{3}{2}y + y^{\frac{1}{2}}\right)\mathrm{d}y + \int_0^1 \left(\frac{3}{2}y + y^{\frac{1}{2}}\right)\mathrm{d}y$$

$$= -\underbrace{\left[\frac{3}{4}y^4 - \frac{2}{3}y^{\frac{3}{2}}\right]_0^1}_{\frac{3}{4}-\frac{2}{3}} + \underbrace{\left[\frac{3}{4}y^4 + \frac{2}{3}y^{\frac{3}{2}}\right]_0^1}_{\frac{3}{4}+\frac{2}{3}} = \frac{4}{3}$$

Notice that it would have worked to just use the inverted limits $\int_1^0 \left(\frac{3}{2}y + y^{\frac{1}{2}}\right)\mathrm{d}y$, which is equivalent to a negative sign, but this can lead to confusion, so do this at your own risk.

CIRCULATION

18) Find the circulation of the field $\mathbf{F} = (x-y)\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$ around the unit circle in the XY plane centred at the origin.

1. **Parametrise the curve**. The unit circle is best parametrised in polar coordinates

$$\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} \text{ with } t \in [0, 2\pi]$$

2. **Find the vector differential of path** $\mathrm{d}\mathbf{r}$:

$$\mathrm{d}\mathbf{r} = \boldsymbol{\tau}\,\mathrm{d}t = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\mathrm{d}t = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}\mathrm{d}t$$

3. **Evaluate the integrand F · dr** along the curve, remember to do the dot product:

$$\mathbf{F}\big(\mathbf{r}(u)\big)\cdot\mathrm{d}\mathbf{r} = \mathbf{F}\big(x(u), y(u), z(u)\big)\cdot\mathrm{d}\mathbf{r} = \begin{pmatrix} x-y \\ x \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}\mathrm{d}t$$

We can do the dot product first, and then substitute $x = \cos t$, $y = \sin t$ and $z = 0$, or the other way around, both give the same result

$$\mathbf{F}\cdot\mathrm{d}\mathbf{r} = (-(x-y)\sin t + x\cos t)\mathrm{d}t = (-(\cos t - \sin t)\sin t + \cos^2 t)\mathrm{d}t = (-\cos t\sin t + \sin^2 t + \cos^2 t)\,\mathrm{d}t = (1 - \cos t \sin t)\,\mathrm{d}t$$

4. **Calculate the integral**:

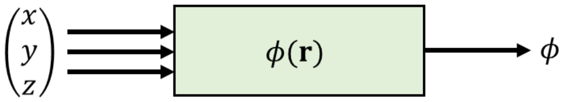
$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^{2\pi}(1 - \cos t\sin t)\mathrm{d}t = \left[t - \frac{1}{2}\sin^2 t\right]_0^{2\pi} = 2\pi$$

5. VECTOR CALCULUS

In this chapter, we look at properties and transformations that can be done on scalar and vector fields.

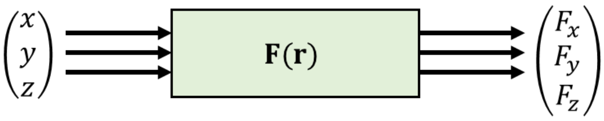
A. SCALAR AND VECTOR FIELDS

Scalar field: $\phi(x, y, z)$



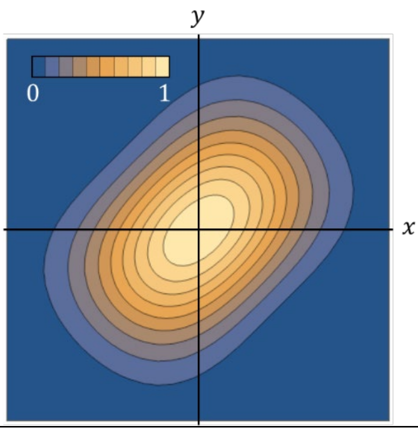
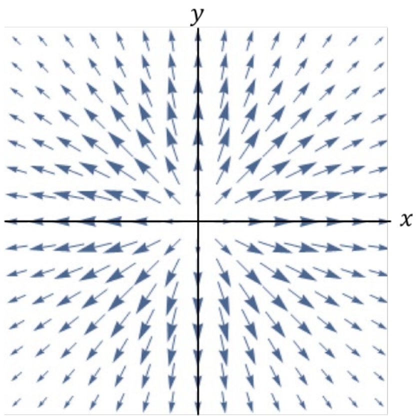
A scalar value assigned continuously to each point in space. **Examples of scalar fields**: temperature in a room, pressure at each point of a fluid, gravitational potential in the solar system, …

Vector field: $\mathbf{F}(x, y, z)$



A vector assigned continuously to each point in space. **Example of vector fields**: velocity vector of a fluid, electric field.

Typical representation of scalar and vector fields: very simple in 2D space.

Scalar field: $T = e^{-x^2+y^2}(1+\sin xy)$	**Vector field:** $\mathbf{F} = (x^2+y^2+1)^{-5/2}(y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}})$
	

Central to all following sections is the vector differential operator ∇ called *del* or *nabla*.

Operator ∇ (called '*del*' or '*nabla*')

$$\nabla \equiv \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial y \end{pmatrix}$$

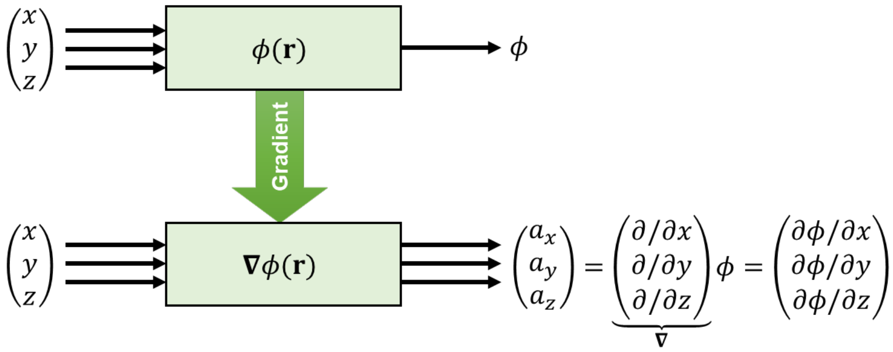
B. GRADIENT

The gradient is a transformation which acts on a scalar field $\phi(\mathbf{r})$, calculated by applying the $\boldsymbol{\nabla}$ operator to the scalar field ϕ:

Gradient:

$$\text{grad } \phi = \boldsymbol{\nabla}\phi(x, y, z) \equiv \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{pmatrix}$$

This operation converts a scalar field into a vector field $\mathbf{a}(\mathbf{r}) = \boldsymbol{\nabla}\phi(\mathbf{r})$.



1) Calculate the gradient of the scalar field $f(x, y, z) = x - xy + z^2$

Solution:

$$\boldsymbol{\nabla} f(x, y, z) \equiv \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \begin{pmatrix} 1 - y \\ -x \\ 2z \end{pmatrix}$$

2) Calculate the gradient of the scalar field $f(x, y) = x^2 - xy$

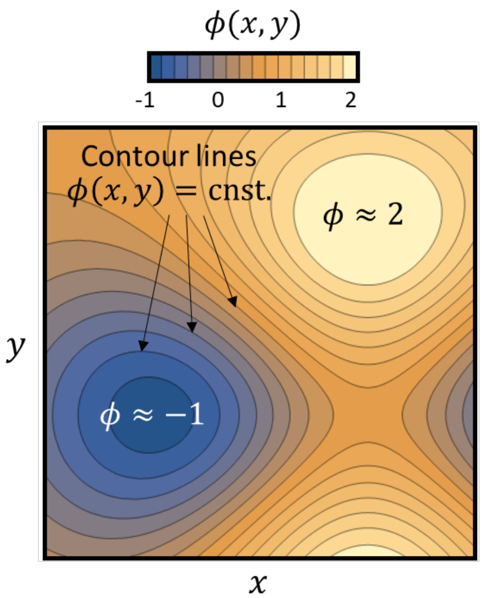
Solution: Note that this case is two-dimensional. The gradient can be defined for N-dimensions simply by adjusting the size of the "nabla" operator to match the number of dimensions.

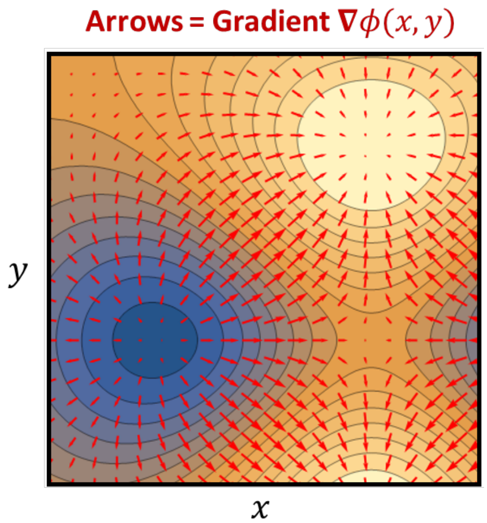
$$\boldsymbol{\nabla} f(x, y, z) \equiv \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} 2x - y \\ -x \end{pmatrix}$$

INTERPRETATION OF THE GRADIENT:

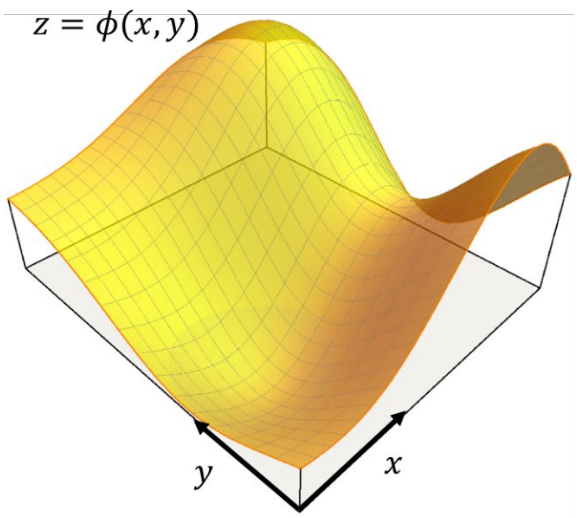
- The gradient $\nabla\phi$ gives us, for each point (x, y, z) in a scalar field ϕ, a **vector** indicating the **direction and magnitude** of the **steepest ascent** along the scalar field.
- Therefore, the gradient is perpendicular to the contour lines of the scalar field ϕ.

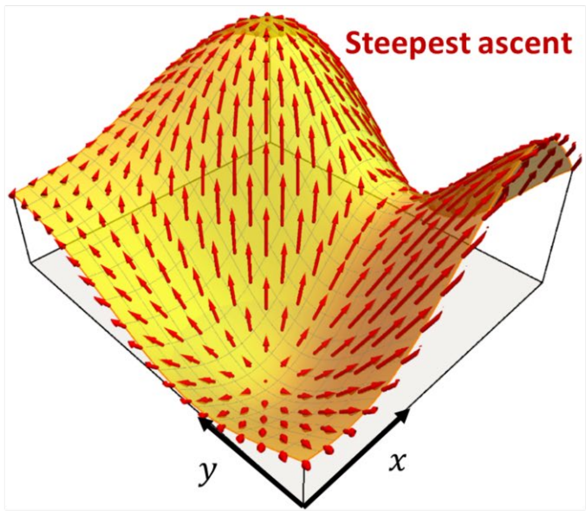
SIMPLE 2-D EXAMPLE:





With 2D fields, we can use a 3D representation in which we use the third dimension to represent the value of the scalar field. Then the gradient is even more intuitive to visualize:





Note how **the gradient always points away from local minima and towards local maxima. The gradient is zero at the stationary points**. A common computational algorithm for finding minima in a function is to follow the opposite direction of the gradient (**method of steepest descent**).

In three dimensions, the gradient is a 3D vector defined in every point in space, pointing along the direction of steepest increase of the scalar field $\phi(\mathbf{r})$.

THE DIRECTIONAL DERIVATIVE

For a scalar field $f(\mathbf{r})$, the directional derivative along a vector $\vec{\mathbf{v}}$ is a scalar that has many different equivalent notations:

$$\nabla_{\vec{\mathbf{v}}} f, \quad D_{\vec{\mathbf{v}}} f, \quad \frac{\partial f}{\partial \vec{\mathbf{v}}}$$

and is defined similarly to a derivative, but moving specifically along the direction $\mathbf{v}$:

$$\nabla_{\vec{\mathbf{v}}} f = \lim_{h \to 0} \frac{f(\mathbf{r} + h\vec{\mathbf{v}}) - f(\mathbf{r})}{h}$$

This is clearly a generalization of partial derivatives, e.g. $\partial f / \partial x = \nabla_{\hat{\mathbf{x}}} f$. In fact, by considering the increase along x, y and z one after the other, we can write the directional derivative in terms of the partial derivatives:

$$\nabla_{\vec{\mathbf{v}}} f = \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y + \frac{\partial f}{\partial z} v_z$$

When $\hat{\mathbf{v}}$ is a unit vector $\nabla_{\hat{\mathbf{v}}} f$ gives the slope of $f(\mathbf{r})$ along the direction $\hat{\mathbf{v}}$.

The dot product between the gradient $\boldsymbol{\nabla} f$ and the vector $\mathbf{v}$ can be used to compute the partial derivative along any vector $\mathbf{v}$. The gradient gives us **all the information** about slopes!

$$\nabla_{\vec{\mathbf{v}}} f = \boldsymbol{\nabla} f \cdot \mathbf{v}$$

This explains why $\boldsymbol{\nabla} f$ is **always orthogonal to contour lines**, along which $f(\mathbf{r})$ is constant.

3) Find the slope of the function $f(x, y) = ye^{x+y}$ in the direction of $\mathbf{u} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}}$ at the location $(x, y) = (1,1)$.

The gradient is:

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(ye^{x+y}, (1+y)e^{x+y}\right)$$

The unit vector in the direction of $\mathbf{u}$ is:

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{(2,1)}{\sqrt{2^2 + 1^2}} = \left(2/\sqrt{5}, 1/\sqrt{5}\right)$$

Therefore:

$$\text{Slope} = \boldsymbol{\nabla} f \cdot \hat{\mathbf{v}} = \left(ye^{x+y}\right)\frac{2}{\sqrt{5}} + (1+y)e^{x+y}\frac{1}{\sqrt{5}}$$

This is the slope in the $\mathbf{u}$ direction at every point in space!

We are asked the slope at $(x, y) = (1,1)$, which is:

$$\text{Slope} = \frac{4e^2}{\sqrt{5}}$$

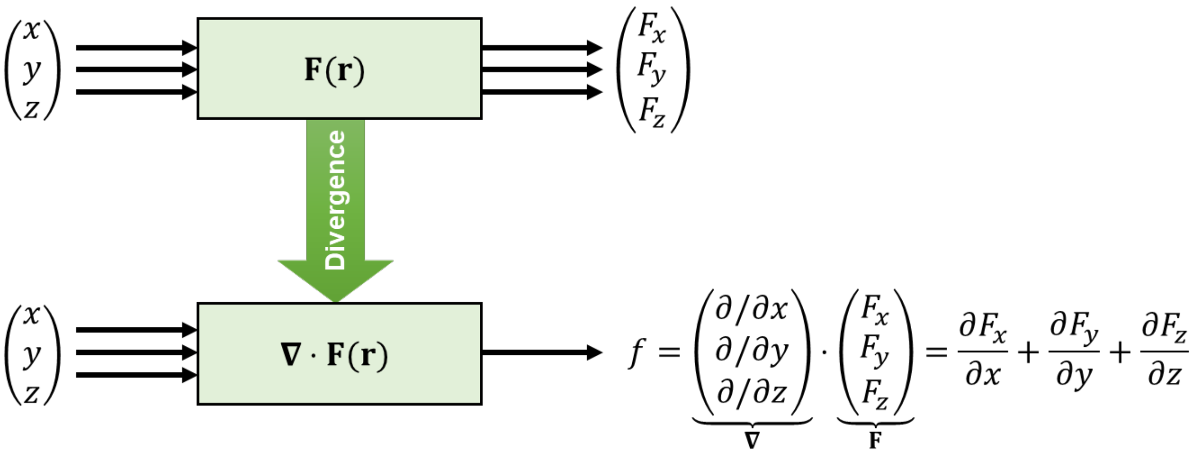
C. DIVERGENCE

The divergence is a transformation which acts on a vector field $\mathbf{F}(\mathbf{r})$. It is calculated in rectangular coordinates by applying the dot product between the $\boldsymbol{\nabla}$ operator and the vector field $\mathbf{F}(\mathbf{r})$:

Divergence:

$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F}(x,y,z) \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

The result is a scalar field. Hence the divergence converts the vector field $\mathbf{F}(\mathbf{r})$ into a scalar field $f(\mathbf{r})$:



$$f = \underbrace{\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}}_{\boldsymbol{\nabla}} \cdot \underbrace{\begin{pmatrix}F_x\\ F_y\\ F_z\end{pmatrix}}_{\mathbf{F}} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

4) Calculate the divergence of the vector field $\mathbf{v}(x,y,z) = \left(3y, 6y - xe^y, \sqrt{xz}\right)$

Solution:

$$\boldsymbol{\nabla}\cdot\mathbf{v}(x,y,z) \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 + (6 - xe^y) + \left(\frac{x}{2\sqrt{xz}}\right)$$

5) Calculate the divergence of the vector field $\mathbf{v}(x,y) = (y^2 - x^3 y)\,\hat{\mathbf{x}} + (6y - 3x^2 y)\,\hat{\mathbf{y}}$

Solution: This is an example in two dimensions:

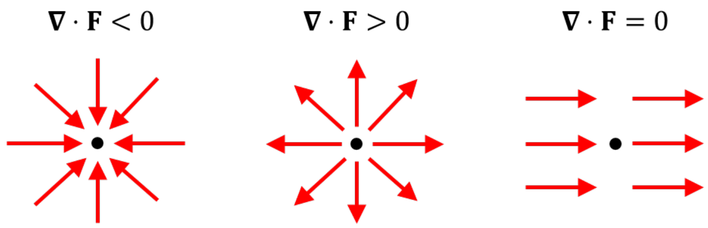
$$\boldsymbol{\nabla}\cdot\mathbf{v}(x,y) \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -3x^2 y + 6 - 3x^2 = 6 - 3x^2(y+1)$$

INTERPRETATION OF DIVERGENCE

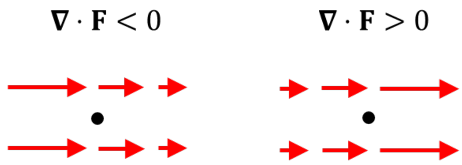
The divergence $\nabla \cdot \mathbf{F}$ gives us, for each point (x, y, z) in a vector field $\mathbf{a}$, a scalar value indicating whether there is a net flux of $\mathbf{a}$ out of (positive) or into (negative) a small volume surrounding the point.

- **Sources** of field lines have $\nabla \cdot \mathbf{F} > 0$
- **Sinks** of field lines have $\nabla \cdot \mathbf{F} < 0$

For a vector field $\mathbf{v}(x, y, z)$ describing the local velocity at any point in a fluid, the divergence $\nabla \cdot \mathbf{v}$ tells us whether the density of the liquid is increasing or decreasing at each point (or whether fluid is appearing or disappearing to keep the density constant).



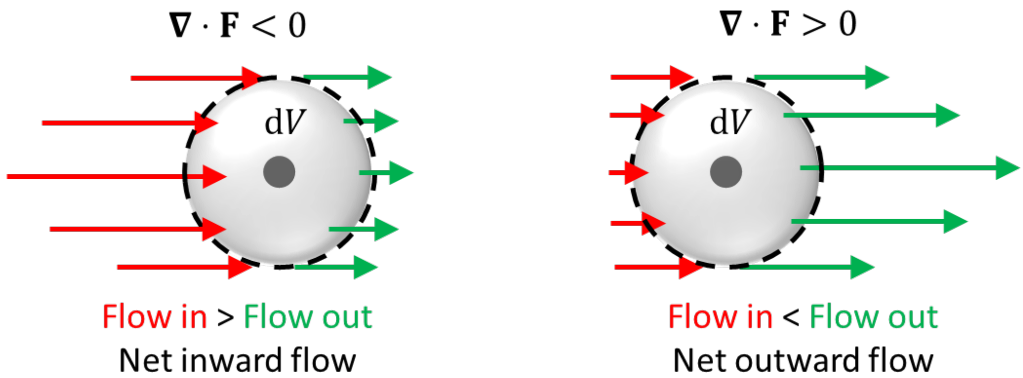
The visual examples above are clear-cut cases. You can still have a positive (negative) divergence if, on average, the strength of the field lines coming out of the point is greater (smaller) than those coming out. Examples:



DEFINITION OF DIVERGENCE IN TERMS OF FLUX:

The divergence is **defined** as **the flux surface integral out of the closed surface enclosing a differential volume element** surrounding each point, normalized by the volume:

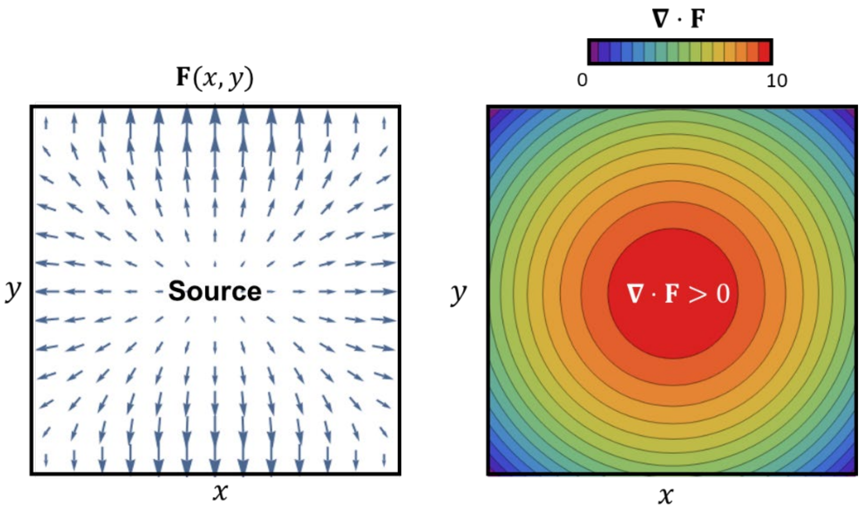
$$\nabla \cdot \mathbf{F} = \lim_{V \to 0} \frac{1}{V} \oint\!\!\!\!\oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$



VISUAL EXAMPLES OF DIVERGENCE IN 2D

Example 1:

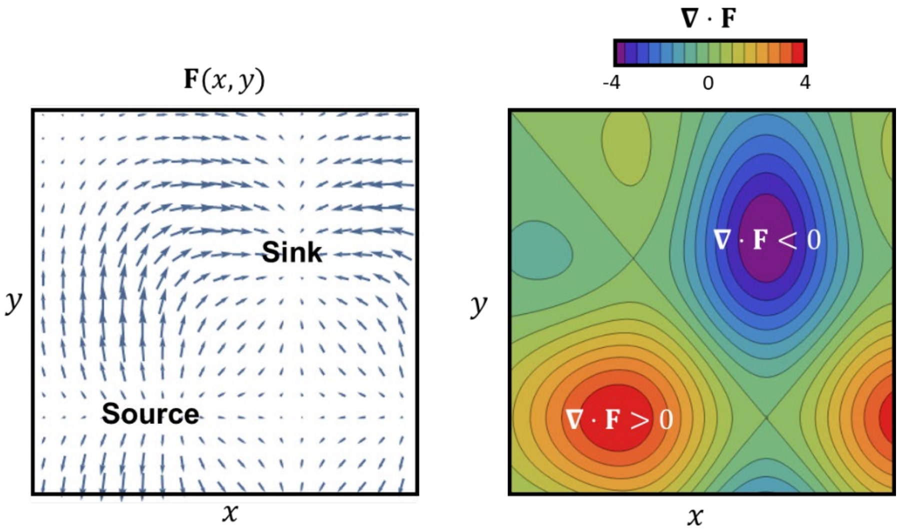
$$\mathbf{F}(x,y) = (4x - xy^2)\hat{\mathbf{x}} + (6y - x^2y)\hat{\mathbf{y}} \quad \rightarrow \quad \nabla \cdot \mathbf{F} = 10 - x^2 - y^2$$



Example 2:

$$\mathbf{F}(x,y) = \frac{1}{2}e^{-\frac{(x^2+y^2)}{4}}\left[x(-4 + x^2 - y^4)\hat{\mathbf{x}} + y(4 + x^4 - y^2)\hat{\mathbf{y}}\right]$$

$$\rightarrow \quad \nabla \cdot \mathbf{F} = \frac{1}{4}e^{-\frac{(x^2+y^2)}{4}}\left(-x^4(-1 + y^2) - y^2(10 + y^2) + x^2(10 + y^4)\right)$$

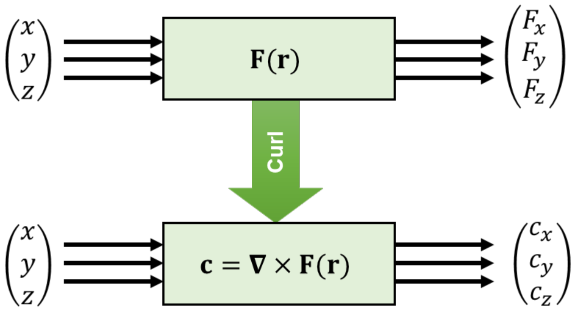


D. CURL

The curl is a transformation which acts on a vector field $\mathbf{a}(\mathbf{r})$. It is calculated in rectangular coordinates by applying the cross product between the $\boldsymbol{\nabla}$ operator and the vector field $\mathbf{a}(\mathbf{r})$:

$$\text{curl } \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}}$$

The result is a vector. Hence, the curl transforms the vector field $\mathbf{F}(\mathbf{r})$ into another vector field $\mathbf{b}(\mathbf{r})$:



$$\begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} = \underbrace{\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}}_{\boldsymbol{\nabla}} \times \underbrace{\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}}_{\mathbf{F}} = \begin{pmatrix} \dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z} \\ \dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x} \\ \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y} \end{pmatrix}$$

6) Calculate the curl of the vector field $\mathbf{a}(x, y, z) = x^2y^2z^2\hat{\mathbf{x}} + y^2z^2\hat{\mathbf{y}} + x^2z^2\hat{\mathbf{z}}$

Solution:

$$\boldsymbol{\nabla} \times \mathbf{a} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$= (0 - 2y^2z)\hat{\mathbf{x}} + (2x^2y^2z - 2xz^2)\hat{\mathbf{y}} + (0 - 2x^2yz^2)\hat{\mathbf{z}}$$

Note: “Scalar” curl for 2D fields.

For two-dimensional fields, we can assume that the field is invariant in the z-direction ($\frac{\partial}{\partial z} = 0$) and that its z-component is zero, so that:

$$\boldsymbol{\nabla} \times \mathbf{F}(x, y) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ F_x & F_y & 0 \end{vmatrix} = \hat{\mathbf{z}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

So, **for 2D fields**, we can see that *the curl may, if we want, be interpreted as a scalar field* (because it always points along z so the direction is not giving us any information).

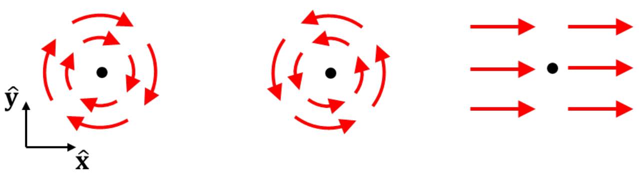
INTERPRETATION OF CURL

For a vector field $\mathbf{v}(x, y, z)$ describing the local velocity at any point in a fluid, the curl $\nabla \times \mathbf{v}$ gives us, for each point (x, y, z), a **vector** value indicating the angular velocity of the fluid in the neighbourhood of that point.

If a small paddle wheel were placed at various points in the fluid then it would tend to rotate in regions where $||\nabla \times \mathbf{v}|| \neq 0$, rotating in a direction such that the right-hand-rule points in the direction of $\nabla \times \mathbf{v}$, and rotating faster for larger values of $||\nabla \times \mathbf{v}||$.

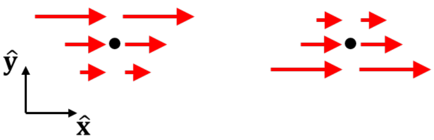
To visualize it in 2D, let's focus only on the z-component of the curl:

$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} < 0$ $\quad$ $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} > 0$ $\quad$ $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} = 0$



The visual examples above are the clearest cases. But you can still have a positive (negative) curl if, on average, the strength of the field lines "pushing anticlockwise" are greater (smaller) than those "pushing clockwise". Examples:

$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} < 0$ $\quad$ $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} > 0$

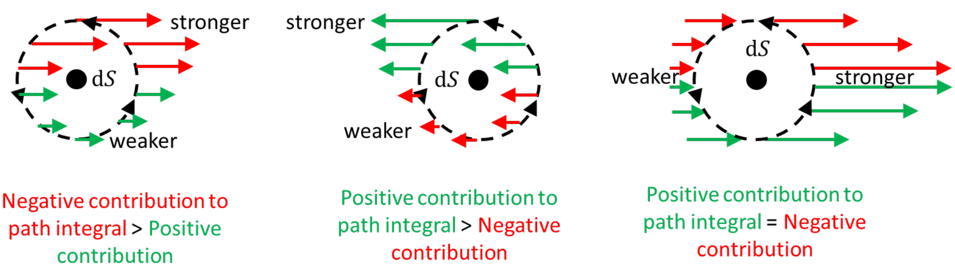


DEFINITION OF CURL IN TERMS OF CIRCULATION:

The component of the curl along a certain direction $\hat{\mathbf{n}}$ is **defined** as **the circulation line integral on a closed path around a differential area element** normal to $\hat{\mathbf{n}}$ surrounding each point, normalized by the area[1]. The integration sense is performed in the direction given by the right-hand rule with $\hat{\mathbf{n}}$:

$$\boxed{(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}}$$

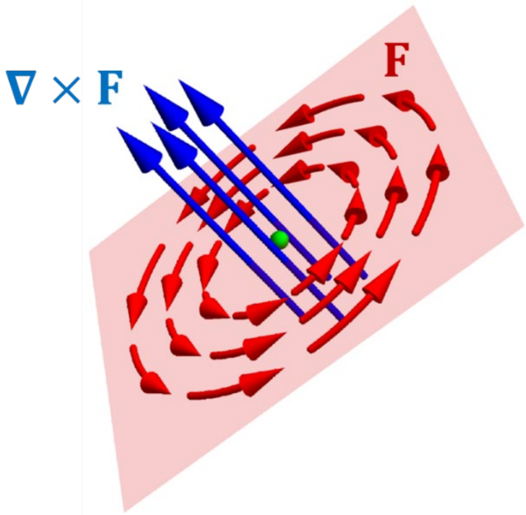
$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} < 0$ $\quad$ $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} > 0$ $\quad$ $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} = 0$



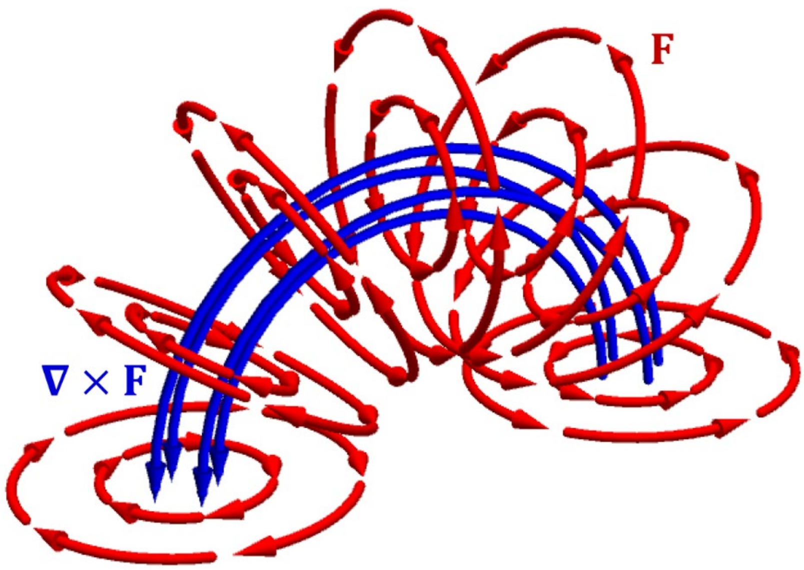
[1] An elegant way to define the curl in all directions simultaneously is $\nabla \times \mathbf{F} = \lim_{V \to 0} \frac{1}{V} \oiint_S \mathrm{d}\mathbf{S} \times \mathbf{F}$

REMEMBER THAT CURL IS A VECTOR DEFINED IN 3D

Previous figures where 2D diagrams. In practice, the curl (unlike the divergence) is a vector. The curl points in the direction perpendicular to the rotation following the right-hand rule.



Both $\mathbf{F}$ and $\mathbf{c} = \nabla \times \mathbf{F}$ are vector fields. They are both defined at every point in space $\mathbf{F}(\mathbf{r})$ and $\mathbf{c}(\mathbf{r})$. Plotting them together can be very complicated and beautiful:



This figure reminds us of the magnetic field created by a bent coil of current-carrying wire. That is not a coincidence, as the **magnetic field** is given as the curl of the **current density** vector field.

Many natural laws are most simple and elegant when they are expressed in terms of gradients, divergences and curls. For example, Maxwell's equations for electromagnetism are all defined in terms of the curl and the divergence of the electric and magnetic fields.

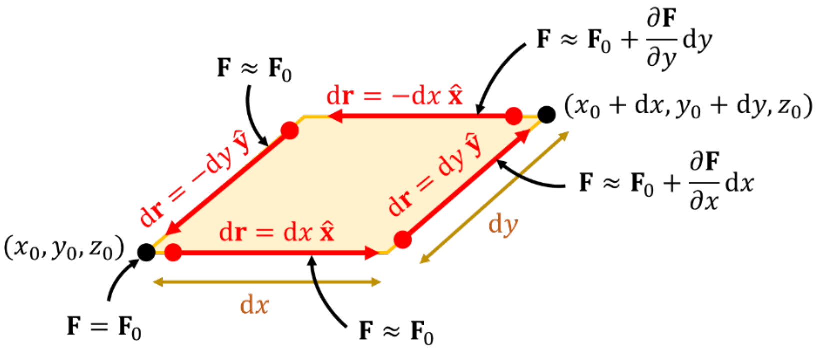
E. DERIVATION OF DIVERGENCE AND CURL FROM THEIR DEFINITION

DERIVATION OF THE CURL FROM ITS DEFINITION

Let's do only the z-component of the curl (the procedure is identical for the other components, just rotate your axes). The z-component of the curl is defined as the circulation of a field $\mathbf{F}$ along a path $P(S)$ which surrounds a small area S perpendicular to $\hat{\mathbf{z}}$, in the limit when the area goes to zero. Therefore, let's consider the area to be a square-shaped differential area $\mathrm{d}S$ of sides $\mathrm{d}x$ and $\mathrm{d}y$ parallel to the XY plane, and let's call the square path surrounding it $P(\mathrm{d}S)$. Then we may say that the circulation around this tiny curve is a differential circulation $\mathrm{d}C$, which divided by $\mathrm{d}S$ gives us the z-component of the curl.

$$(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{z}} = \lim_{A\to 0}\frac{1}{A}\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \frac{1}{\mathrm{d}S}\underbrace{\oint_{P(dS)}\mathbf{F}\cdot\mathrm{d}\mathbf{r}}_{\mathrm{d}C} = \frac{1}{\mathrm{d}S}\mathrm{d}C$$

The circulation on this square path can be calculated as four different line integrals added together. What is the value of the field at each location? At one corner of this square we have a vector field $\mathbf{F}(x_0, y_0, z_0)$ which we may call $\mathbf{F}_0$. The field throughout this differential rectangle will be almost constant, however let's expand it to **first order** by considering differential changes: $\mathbf{F}(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) \approx \mathbf{F}(x_0,y_0,z_0) + \mathrm{d}\mathbf{F}(\Delta x,\Delta y,\Delta z) = \mathbf{F}_0 + \frac{\partial\mathbf{F}}{\partial x}\Delta x + \frac{\partial\mathbf{F}}{\partial y}\Delta y + \frac{\partial\mathbf{F}}{\partial z}\Delta z$. When doing the line integral, we may consider the vector field to be constant along each side, so the line integral is simply a multiplication of the length times the field in the direction of the line (due to the dot product). Then we can approximate the field at each side of the rectangle as follows:



The circulation is then given by the dot product of the field with the vector length of each side:

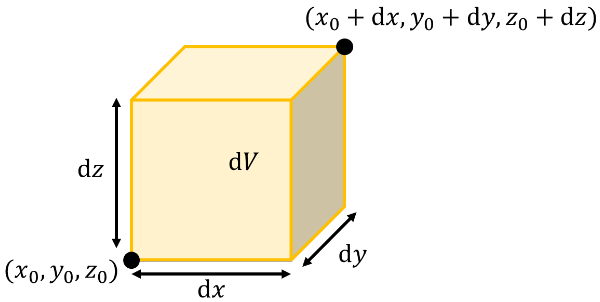
$$\begin{aligned}
\oint \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \mathrm{d}C &= (\mathbf{F}_0)\cdot(\mathrm{d}x\,\hat{\mathbf{x}}) + \left(\mathbf{F}_0 + \frac{\partial\mathbf{F}}{\partial x}\mathrm{d}x\right)\cdot(\mathrm{d}y\,\hat{\mathbf{y}}) + \left(\mathbf{F}_0 + \frac{\partial\mathbf{F}}{\partial y}\mathrm{d}y\right)\cdot(-\mathrm{d}x\,\hat{\mathbf{x}}) + (\mathbf{F}_0)\cdot(-\mathrm{d}y\,\hat{\mathbf{y}}) \\
&= F_{0x}\mathrm{d}x + \left(F_{0y} + \frac{\partial F_y}{\partial x}\mathrm{d}x\right)\mathrm{d}y + \left(-F_{0x} - \frac{\partial F_x}{\partial y}\mathrm{d}y\right)\mathrm{d}x + \left(-F_{0y}\right)\mathrm{d}y \\
&= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Which results in the z-component of the curl after dividing by $\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y$. An identical result is obtained if we consider other orientations of the surface to find other components of the curl.

DERIVATION OF DIVERGENCE FROM ITS DEFINITION

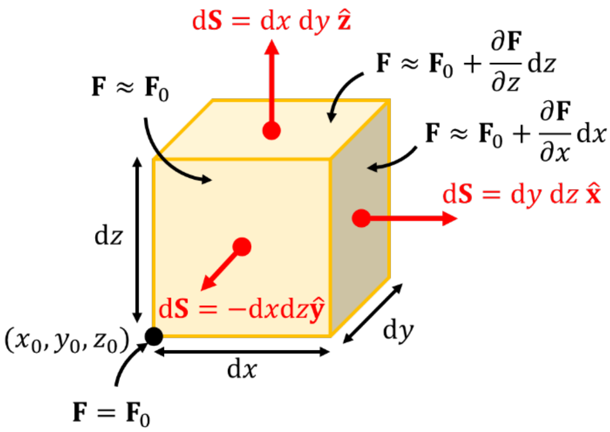
$$\boldsymbol{\nabla}\cdot\mathbf{F} \stackrel{\text{def}}{=} \lim_{V\to 0}\frac{1}{V}\oiint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}$$

Consider a differential cubic volume of sides $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$. The total volume is $\mathrm{d}V$.



Now we may evaluate the flux across the cubic surface (i.e. a sum over each of the six sides of the differential volume). We may call this surface enclosing $\mathrm{d}V$ as $S(\mathrm{d}V)$. The resulting flux on this differential surface can be considered a differential of flux $\mathrm{d}\Phi$.

$$\boldsymbol{\nabla}\cdot\mathbf{F} \stackrel{\text{def}}{=} \frac{1}{\mathrm{d}V}\underbrace{\oiint_{S(\mathrm{d}V)} \mathbf{F}\cdot \mathrm{d}\mathbf{S}}_{\mathrm{d}\Phi} = \frac{1}{\mathrm{d}V}\mathrm{d}\Phi$$



The six fluxes over the six sides can be calculated as in the table below (see diagram above for some intuition). For each of the six differential surfaces, we may consider the field to be constant across the surface, so that the flux is just the product of the field normal to the surface times the surface area.

Equation of surface	Surface normal $\hat{\mathbf{n}}$	Area of surface $\mathrm{d}S$	Vector field on surface $\mathbf{F}$	Flux over surface: $\mathrm{d}\Phi = \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$
$x = x_0 + \mathrm{d}x$	$\hat{\mathbf{x}}$	$\mathrm{d}y\,\mathrm{d}z$	$\mathbf{F}_0 + \frac{\partial \mathbf{F}}{\partial x}\mathrm{d}x$	$\left(F_{0x} + \frac{\partial F_x}{\partial x}\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}z$
$x = x_0$	$-\hat{\mathbf{x}}$	$\mathrm{d}y\,\mathrm{d}z$	$\mathbf{F}_0$	$-(F_{0x})\,\mathrm{d}y\,\mathrm{d}z$
$y = y_0 + \mathrm{d}y$	$\hat{\mathbf{y}}$	$\mathrm{d}x\,\mathrm{d}z$	$\mathbf{F}_0 + \frac{\partial \mathbf{F}}{\partial y}\mathrm{d}y$	$\left(F_{0y} + \frac{\partial F_y}{\partial y}\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}z$
$y = y_0$	$-\hat{\mathbf{y}}$	$\mathrm{d}x\,\mathrm{d}z$	$\mathbf{F}_0$	$-(F_{0y})\,\mathrm{d}x\,\mathrm{d}z$

$z = z_0 + \mathrm{d}x$	$\hat{\mathbf{z}}$	$\mathrm{d}x\,\mathrm{d}y$	$\mathbf{F}_0 + \frac{\partial \mathbf{F}}{\partial z}\mathrm{d}z$	$\left(F_{0z} + \frac{\partial F_z}{\partial z}\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y$
$z = z_0$	$-\hat{\mathbf{z}}$	$\mathrm{d}x\,\mathrm{d}y$	$\mathbf{F}_0$	$-(F_{0z})\,\mathrm{d}x\,\mathrm{d}y$

If we add up the six fluxes across the six surfaces of the differential cube (adding the last column of the table above), all the appearances of $\mathbf{F}_0$ cancel out, and we get a total differential flux equal to:

$$\mathrm{d}\Phi = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Which after dividing by $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ gives us the divergence.

$$\nabla \cdot \mathbf{F} \stackrel{\text{def}}{=} \frac{\mathrm{d}\Phi}{\mathrm{d}V} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

F. VECTOR CALCULUS IDENTITIES:

The following mathematical identities can be checked directly from the definition of the operators.

PROPERTIES OF DIVERGENCE:	**PROPERTIES OF CURL:**
$\nabla \cdot (\mathbf{a} + \mathbf{b}) = \nabla \cdot \mathbf{a} + \nabla \cdot \mathbf{b}$ $\nabla \cdot (\phi\mathbf{a}) = \phi(\nabla \cdot \mathbf{a}) + \mathbf{a} \cdot (\nabla\phi)$	$\nabla \times (\mathbf{a} + \mathbf{b}) = \nabla \times \mathbf{a} + \nabla \times \mathbf{b}$ $\nabla \times (\phi\mathbf{a}) = \phi(\nabla \times \mathbf{a}) - \mathbf{a} \times (\nabla\phi)$

COMBINING CURL, DIVERGENCE AND GRADIENT

- **Curl of a gradient is always zero:** $\nabla \times (\nabla\phi) = \mathrm{curl}(\mathrm{grad}(\phi)) = \mathbf{0}$
- **Divergence of a curl is always zero:** $\nabla \cdot (\nabla \times \mathbf{F}) = \mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0$

- **Divergence of gradient:** $\nabla \cdot \nabla\phi = \mathrm{div}(\mathrm{grad}(\phi)) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \equiv \nabla^2\phi$
  - $\nabla^2\phi$ is called **Laplacian operator**
- **Curl of curl:** $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$ where $\nabla^2\mathbf{F}$ is called the vector Laplacian (Laplacian applied to each component).

Do not worry about memorizing all these identities. I will provide them in the exam if they are needed. The only identities you certainly must remember because they are simple and used in physical arguments (see later) are the ones in the yellow box above.

G. POTENTIALS AND CONSERVATIVE/IRROTATIONAL/SOLENOIDAL FIELDS

CONSERVATIVE FIELDS AND POTENTIALS

Many vector fields in physics (gravitational field, electrostatic field, …) are associated with a corresponding potential energy (gravitational potential energy, electrostatic potential energy, …). These are called **fields derived from a potential**, or **conservative fields**: The field $\mathbf{F}$ points in the direction of steepest descent of a scalar potential ϕ, and its magnitude depends on the rate of change of the potential. This is mathematically described by a gradient.

$$\text{Conservative field} \iff \mathbf{F} = -\nabla\phi$$

The **line integral** of a conservative field has the special property that it is independent of the path chosen, it only depends on the potential difference between the two ends of the path:

$$\text{Conservative field} \iff \int_A^B \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \phi_A - \phi_B \quad \text{for every path joining A and B}$$

Example: work done by gravitational force along a path $W = \int_A^B \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \phi_A - \phi_B$.

Proof: $\mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\nabla\phi\cdot \mathrm{d}\mathbf{r} = -\left[\left(\frac{\partial\phi}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial\phi}{\partial y}\right)\mathrm{d}y + \left(\frac{\partial\phi}{\partial x}\right)\mathrm{d}z\right] = -\mathrm{d}\phi$ by recalling the definition of the total differential. Therefore $\mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\mathrm{d}\phi$ so: $\int_A^B \mathbf{F}\cdot \mathrm{d}\mathbf{r} = -\int_A^B \mathrm{d}\phi = -(\phi_B - \phi_A)$

From path independence we can conclude that the circulation on any closed loop must be zero, because the endpoints are at the same potential:

$$\text{Conservative field} \iff \oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0 \quad \text{for every closed path } C$$

FINDING THE POTENTIAL FROM THE FIELD:

We can find the potential at any point (up to an arbitrary constant), given the field, by doing a path integral:

$$\phi(x,y,z) = \phi_O - \int_O^{(x,y,z)} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

We can choose any path. The most convenient to solve analytically is generally $(0,0,0) \to (x,0,0) \to (x,y,0) \to (x,y,z)$ or similar, where only one variable at a time is varying while others are constant.

IRROTATIONAL FIELDS

A field is called irrotational (or **curl-free**) when its rotational is zero:

$$\text{Irrotational field} \iff \boldsymbol{\nabla} \times \mathbf{F} = 0$$

It is easy to prove that all conservative fields are irrotational, because the curl of a gradient is zero (one of the vector identities). But are all irrotational fields conservative? The answer is yes* but only **if** the region in which the field is defined is a "**simply connected region**" (i.e. it has no holes). In that case, every irrotational field is conservative. In summary:

$$\underset{\text{Irrotational field}}{\boldsymbol{\nabla} \times \mathbf{F} = 0} \quad \overset{*}{\Longrightarrow} \atop \Longleftarrow \quad \underset{\text{Conservative field}}{\mathbf{F} = -\nabla\phi} \quad \iff \quad \underset{\text{Path independence}}{\int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \phi_A - \phi_B}$$

SOLENOIDAL FIELDS

A field is called solenoidal (or **divergence-free**) when its divergence is zero:

$$\text{Solenoidal field} \iff \boldsymbol{\nabla} \cdot \mathbf{F} = 0$$

From the intuition we have on the divergence, this means that the field has no isolated sources nor sinks. The field lines are always closed (no open ends). In turn, this means that the surface flux on any closed surface must be zero because any "field line" entering the volume must exit it elsewhere (also see Gauss' theorem).

$$\underset{\text{Solenoidal field}}{\boldsymbol{\nabla} \cdot \mathbf{F} = 0} \quad \iff \quad \begin{matrix}\text{No sources nor sinks} \\ \text{Closed field lines (no open ends)}\end{matrix} \quad \iff \quad \underset{\text{Closed surface}}{\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0}$$

SIGNIFICANCE IN PHYSICS

Real liquid flows are "incompressible", so no change in pressure can happen, and liquid cannot be created or destroyed: thus, real liquid flows are typically modelled as solenoidal $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$.

As far as we know, magnetic fields do not have sources nor sinks (no magnetic monopoles), and so $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ is one of Maxwell's equations of electromagnetism. On the other hand, electric charges act like sources (positive) and sinks (negative) of electric field, and so the electric field is not solenoidal; in fact $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\varepsilon$, where $\rho(\mathbf{r})$ is the charge density distribution.

EXPANSION OF FIELD INTO SOLENOIDAL AND IRROTATIONAL COMPONENTS:

Every field in a simply connected region can be decomposed into a sum of an irrotational and a solenoidal part: $\mathbf{A}(\mathbf{r}) = \mathbf{A}_c(\mathbf{r}) + \mathbf{A}_s(\mathbf{r})$ with $\nabla \times \mathbf{A}_c(\mathbf{r}) = 0$ and $\nabla \cdot \mathbf{A}_s(\mathbf{r}) = 0$.

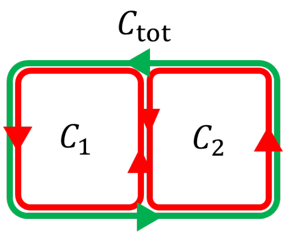
H. STOKES' THEOREM AND DIVERGENCE THEOREM

Optional read: Not in the exam. You will study these two fundamental theorems in 2nd Year.

STOKES THEOREM:

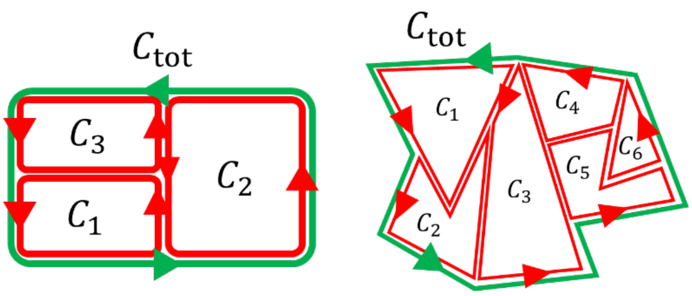
Let's build up the steps needed to arrive at Stokes' theorem:

If we add the circulation of a field around two adjacent paths C_1 and C_2 in the same sense (e.g. anticlockwise), the section of the path at the shared boundary between the two paths is traversed in opposite directions, and so the contributions from that shared boundary to the total line integral will cancel out after the addition (as they are of equal magnitude but opposite sign). The resulting addition of the circulations will therefore be equal to the circulation across a bigger path enclosing both paths, like this:



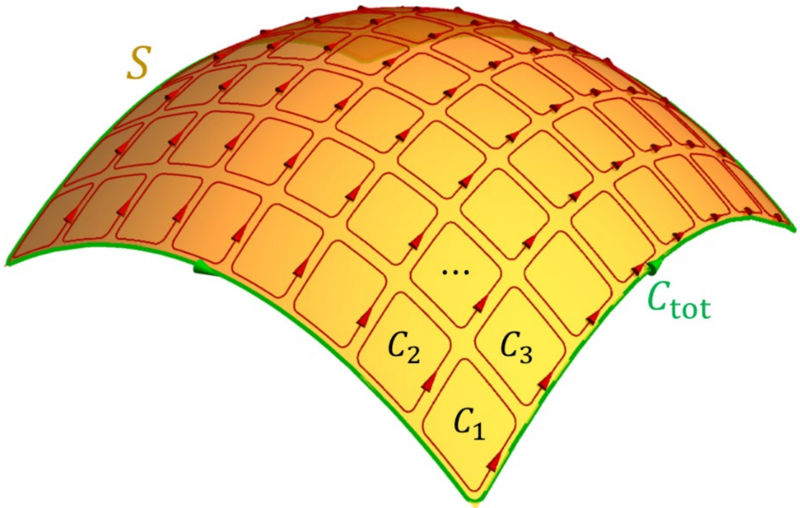
$$\oint_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \oint_{C_{\text{tot}}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

This argument can be extended to an arbitrary number of neighbouring paths, as long as they are all defined in the same direction, and as long as they do not leave an gaps between them (in maths speech, they define a "simply connected region"). The shared inner boundaries are always traversed exactly twice, once in each of the opposite directions, and hence they all cancel out, leaving only the circulation around their common outer boundary:

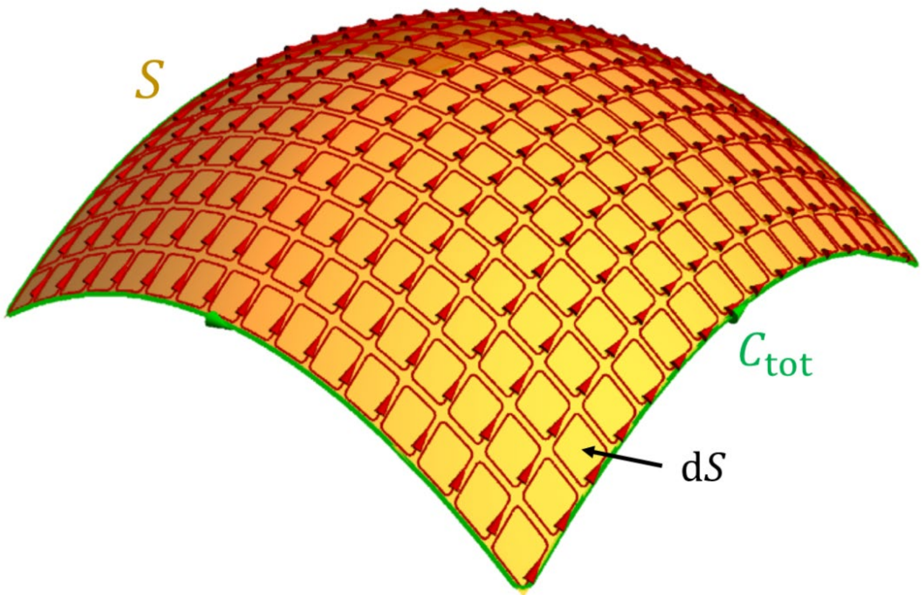


$$\sum_{i=1}^{N} \left(\oint_{C_i} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \right) = \oint_{C_{\text{tot}}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

This argument is not limited to planar 2D space. It works for any collection of paths in three dimensions, covering an arbitrary "simply connected" surface S with no gaps, and with an exterior boundary curve C_{tot}.



The first fundamental leap towards Stokes' theorem is to take the limit when these areas become infinitely small, i.e. exactly equivalent to division of a surface into differentials of surface $\mathrm{d}S$:



In each differential of surface $\mathrm{d}S$ we need to evaluate the differential circulation $\mathrm{d}C \equiv \lim_{A\to 0} \oint_{C_i} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ around its differential boundary, and then we need to add them together. Since they are differentials, we need to add them together via a surface integral $\sum_{i=1}^{N} C_i \to \iint_S \mathrm{d}C$. But following the logic above, the total surface integral, sum of all the circulations, will still be equal to the circulation around the exterior path C_{tot}, because all the shared boundaries cancel out, even after taking the limit:

$$\iint_S \mathrm{d}C = \oint_{C_{\text{tot}}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \qquad (1)$$

The second fundamental leap is to notice that the differential circulation around a differential surface $\mathrm{d}S$ normal to a vector $\hat{\mathbf{n}}$ (normal to the surface) is **exactly the definition of the curl** projected in the direction $\hat{\mathbf{n}}$ at each point (multiplied by the differential area $\mathrm{d}S$). Let's clarify this: the curl in a given direction is defined as the circulation around an area (we can call the area S and the path around it

$C(S)$) when the area S tends to zero. This definition can be re-interpreted in terms of ratios of differentials.

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \lim_{S \to 0} \frac{1}{S} \oint_{C(S)} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{1}{\mathrm{d}S} \underbrace{\oint_{C(\mathrm{d}S)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}}_{\mathrm{d}C} = \frac{1}{\mathrm{d}S}\, \mathrm{d}C$$

Hence, the curl in a given direction $\hat{\mathbf{n}}$ is the ratio between the differential circulation $\mathrm{d}C$ around a differential surface $\mathrm{d}S$ normal to $\hat{\mathbf{n}}$ and the differential surface $\mathrm{d}S$ itself:

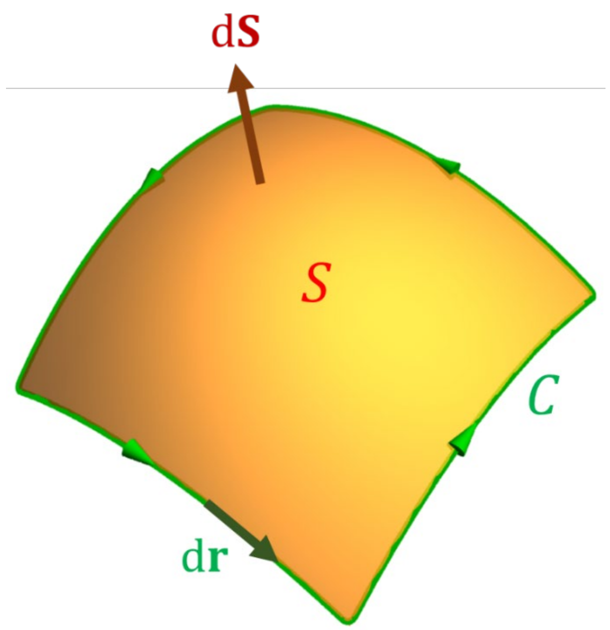
$$\boxed{(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \frac{\mathrm{d}C}{\mathrm{d}S}}$$

We can rewrite this as $\mathrm{d}C = (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S = (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$, following the usual definition $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\, \mathrm{d}S$, which we can substitute into Eq. (1) above to arrive at Stokes' theorem:

STOKES THEOREM

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

According this theorem, S is **any** simply connected surface in three dimensions which is bounded by the closed path C. $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is the circulation of any vector field along the closed path C, while $\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$ is the flux of the curl of the same vector field through the surface S.



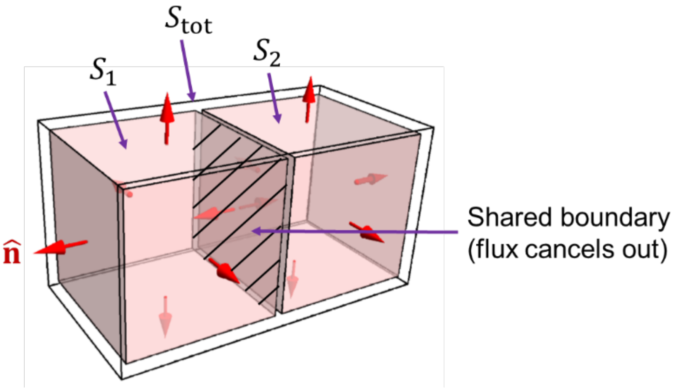
With Stokes' theorem, we can convert a surface integral into a contour line integral, and vice-versa, as long as we know the vector field at the contour, and its curl at a surface enclosed by the contour.

The calculation of the flux always has a sign ambiguity. We must always take the orientation of the normal to the surface in accordance to the right-hand rule, when the fingers of the hand are curling in the direction of the closed path C, the thumb will point in the direction of $\mathbf{dS}$.

DIVERGENCE THEOREM (ALSO KNOWN AS GAUSS' THEOREM OR OSTROGRADSKY'S THEOREM)

The explanation for the divergence's theorem is very similar to that of Stokes'. This is no coincidence, because both theorems are in reality different aspects of one same unified generalised theorem.

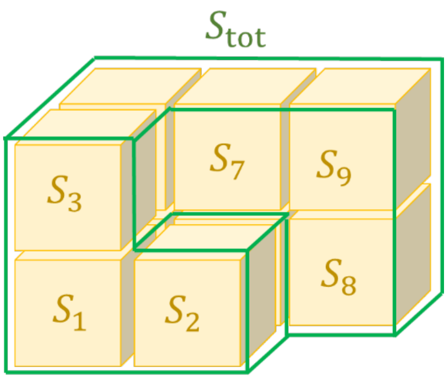
We start by consider what happens when we add the outward flux of a field out of two adjacent **closed** surfaces S_1 and S_2 (in this case, two touching cubes):

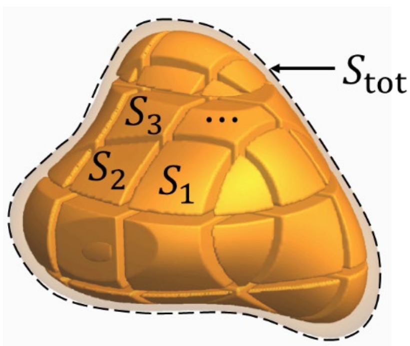


$$\oiint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \oiint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \oiint_{S_{\text{tot}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

We can see that the surface of the cubes where they both come into contact (the shared boundary between the two closed surfaces) will be integrated twice when adding the two flux integrals, but with an opposite normal vector $\hat{\mathbf{n}}$ (we calculate the flux outwards from each closed surface), and so the contributions from that shared surface to the total flux will cancel out after the addition of the two fluxes (as they are of equal magnitude but opposite sign). The resulting addition of the flux will therefore be equal to the flux outwards from the bigger surface S_{tot} that encloses both surfaces.

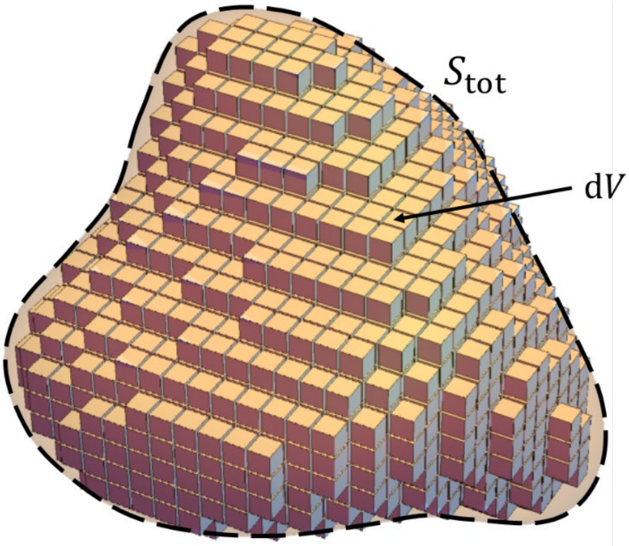
This argument can be extended to an arbitrary number of neighbouring volumes as long as they don't leave gaps between them. The sum of the fluxes over many closed surfaces will be equal to the flux out of the total volume they cover. This is because the flux over their shared inner surfaces is always integrated exactly twice, once with each of the two opposite directions of the normal vector, and hence all inner surface fluxes will cancel out, leaving only the flux around their outer common surface boundary. This is as true for cubes as for any other arbitrary shapes which fit together without gaps:





$$\sum_{i=1}^{N} \left(\oiint_{S_i} \mathbf{F} \cdot \mathrm{d}\mathbf{S} \right) = \oiint_{S_{\text{tot}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

The first fundamental leap towards the divergence theorem is to take the limit when all the surfaces S_i become infinitely small, surrounding differentials of volume $\mathrm{d}V$:



Now, in each differential of volume $\mathrm{d}V$ we need to evaluate the differential flux $\mathrm{d}\Phi \equiv \lim_{A\to 0} \oiint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}$ around its differential outer surface, and then we need to add them together. Since they are differentials, we need to add them together via a volume integral. But following the logic above, the total volume integral, sum of all the fluxes for each $\mathrm{d}V$, will be equal to the total flux across the exterior surface S_{tot}, because all the inner shared boundary surfaces cancel out, even after taking the limit:

$$\iiint_V \mathrm{d}\Phi = \oiint_{S_{\text{tot}}} \mathbf{F}\cdot \mathrm{d}\mathbf{S} \tag{2}$$

The second fundamental leap is to notice that the differential flux around a differential volume element $\mathrm{d}V$ is **exactly the definition of the divergence** at each point (multiplied by the differential volume $\mathrm{d}V$). Let's explain this: the divergence at any point is defined as the flux out of a surface enclosing a volume (we can call the volume V and the surface which encloses it $S(V)$) divided by the volume when the volume V tends to zero. This definition can be re-interpreted in terms of ratios of differentials.

$$\nabla\cdot\mathbf{F} = \lim_{V\to 0}\frac{1}{V}\oiint_{S(V)}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \frac{1}{\mathrm{d}V}\underbrace{\oiint_{S(\mathrm{d}V)}\mathbf{F}\cdot\mathrm{d}\mathbf{S}}_{\mathrm{d}\Phi} = \frac{1}{\mathrm{d}V}\mathrm{d}\Phi$$

Hence, the divergence is the ratio between the differential flux $\mathrm{d}\Phi$ out of a differential volume $\mathrm{d}V$ and the differential volume itself:

$$\boxed{\nabla\cdot\mathbf{F} = \frac{\mathrm{d}\Phi}{\mathrm{d}V}}$$

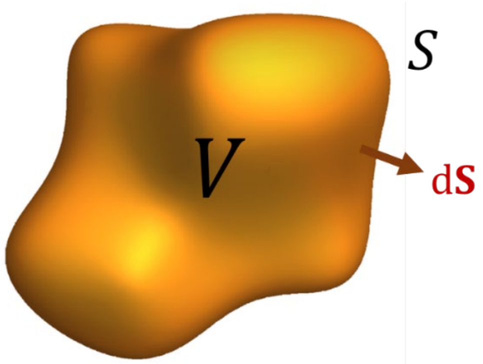
We can rewrite this as $\mathrm{d}\Phi = \nabla\cdot\mathbf{F}\,\mathrm{d}V$, which we can substitute into Eq. (2) above to finally arrive at the divergence theorem:

DIVERGENCE THEOREM

$$\iiint_V (\nabla \cdot \mathbf{F})\, \mathrm{d}V = \oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

In this theorem, V is a simply connected volume in three dimensions, which is bounded by a closed surface S. The term $\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ is the flux of any vector field outwards from the closed surface S, while $\iiint_V (\nabla \cdot \mathbf{F})\, \mathrm{d}V$ is the volume integral of the divergence of the same vector field on the inside of the entire volume V. The orientation of $\mathrm{d}\mathbf{S}$ for the surface flux integral must always be taken pointing towards the outside of the volume.

Intuitively, the theorem states that **the sum of all sources (with sinks regarded as negative sources) gives the net flux out of a region**.



With the divergence theorem we can convert a volume integral into a flux surface integral, and vice-versa, as long as we know the vector field at the surface, and its divergence inside the volume.

GREEN'S THEOREM:

Stokes' and the divergence theorems can be applied to two-dimensional fields in the XY plane. For Stokes' theorem, the surface integral and circulation become a double integral and a circulation in the XY plane.

$$\iint_A \overbrace{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}^{\mathrm{curl}\,\mathbf{F}} \mathrm{d}x\, \mathrm{d}y \;= \oint_C F_x \mathrm{d}x + F_y \mathrm{d}y$$

For the divergence theorem, the closed surface flux becomes a "2D flux" outside of a closed curve (a line integral), and the volume integral becomes a double integral, both in the XY plane:

$$\iint_A \overbrace{\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)}^{\mathrm{div}\,\mathbf{F}} \mathrm{d}x\, \mathrm{d}y = \oint_C F_x \mathrm{d}y - F_y \mathrm{d}x$$

Notice that both equations are equivalent if we swap $F_y \to F_x$ and $F_x \to -F_y$. In fact, F_x and F_y could be anything, not necessarily components of a field. They can be **<u>any two independent functions</u>** $F_x = L(x,y)$ and $F_y = M(x,y)$. With that substitution, we arrive at the so-called Green's Theorem.

I. CONTINUITY EQUATION

Optional read: Not in the exam. You will study this, with examples, in 2nd Year.

Now we are going to add time as a parameter. Very often, vector fields are not static $\mathbf{J}(\mathbf{r})$, but instead change with time $\mathbf{J}(\mathbf{r}, t)$.

Let's assume that the vector field $\mathbf{J}(\mathbf{r}, t)$ represents the flow of some quantity which is conserved. We could use any conserved quantity (e.g. mass, energy, charge) but for the sake of example, let's use the **charge** flowing per unit area and per unit time at each point, also called the current density. Then, whenever there is a non-zero flux of this vector across of a **closed** surface, this necessarily means that the total amount of charge must be changing inside the volume, due to conservation of charge. This will be a change with time. It can be written mathematically as:

$$\frac{\mathrm{d}Q_V}{\mathrm{d}t} = -\oint\!\!\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$$

where the variable Q_V represents the total amount of charge inside the volume V enclosed by S. The above equation reads: "*the change in time of the total amount Q_V of a conserved quantity in a given volume V must be equal to the amount of that quantity flowing into the volume per unit time (that is, the outward flux on the enclosing surface S with a minus sign)*".

Since the variable Q_V is the total charge inside the volume, we can write it as a volume integral of the charge density ρ (amount of charge per unit volume):

$$Q_V = \iiint_V \rho \, \mathrm{d}V$$

Hence the continuity equation can be written as:

CONTINUITY CONDITION (INTEGRAL FORM)

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \, \mathrm{d}V = -\oint\!\!\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$$

If we now apply the divergence theorem to the right-hand side:

$$\oint\!\!\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \iiint_V \nabla \cdot \mathbf{J} \, \mathrm{d}V$$

We arrive at (we can take the time derivative inside the integral due to linearity):

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \, \mathrm{d}V = \iiint_V \frac{\mathrm{d}\rho}{\mathrm{d}t} \, \mathrm{d}V = -\iiint_V \nabla \cdot \mathbf{J} \, \mathrm{d}V$$

Since the equation must be true for any volume V, we can make the volume infinitesimally small, and so the integrands must be equal at every point, resulting in the elegant continuity condition:

CONTINUITY CONDITION (DIFFERENTIAL FORM)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\nabla \cdot \mathbf{J}$$

PROBLEMS

7) Find the gradient of the following scalar fields:

a) $\phi = 1$

b) $\phi = x + y + z$

c) $\phi = e^{-xyz}$

d) $\phi = e^x(y^2 + \ln zy)$

Solution:

$$\nabla\phi(x,y,z) \equiv \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{pmatrix}$$

a) $\nabla\phi = (0,0,0)$

b) $\nabla\phi = (1,1,1)$

c) $\nabla\phi = (-yze^{-xyz}, -xze^{-xyz}, -xye^{-xyz})$

d) $\nabla\phi = \left((y^2 + \ln zy)e^x, 2e^x y + \frac{e^x}{y}, \frac{e^x}{z}\right)$

8) Find the slope of $\phi = x^3y + yz^2 + z$ in the direction of the following vector $\mathbf{u} = (1,1,1)$ at the origin.

Solution: First find the gradient:

$$\nabla\phi(x,y,z) \equiv \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial z \end{pmatrix} = \begin{pmatrix} 3x^2y \\ x^3 + z^2 \\ 2yz + 1 \end{pmatrix}$$

Now find the unit vector in the direction of $\mathbf{u}$

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The slope is equal to the directional derivative in the direction of $\hat{\mathbf{u}}$, which is the dot product between the gradient and the unit vector.

$$\text{Slope} = \nabla\phi \cdot \hat{\mathbf{u}} = \frac{1}{\sqrt{3}}(3x^2y + x^3 + z^2 + 2yz + 1)$$

This is the slope in the $\hat{\mathbf{u}}$ direction at every point in space! We are asked the slope at $(x,y,z) = (0,0,0)$, so we substitute $x = y = z = 0$ (note we must do all the calculations of gradient **before** substituting the specific point, otherwise we could not calculate derivatives):

$$\text{Slope} = \frac{1}{\sqrt{3}}$$

9) Find the divergence and the curl of the following vector fields:

a) $\mathbf{F} = (-y, xy, z)$

b) $\mathbf{F} = (y, e^{xy}, 1)$

c) $\mathbf{F} = 2xe^x y\,\hat{\mathbf{x}} + xy^2 \cos z\,\hat{\mathbf{y}} - (y+z)\hat{\mathbf{z}}$

Calculation of the divergence:

$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

a) $\boldsymbol{\nabla}\cdot\mathbf{F} = 0 + x + 1 = x + 1$

b) $\boldsymbol{\nabla}\cdot\mathbf{F} = 0 + xe^{xy} + 0 = xe^{xy}$

c) $\boldsymbol{\nabla}\cdot\mathbf{F} = 2xye^x + 2ye^x + 2xy\cos z - 1 = 2ye^x(x+1) + 2xy\cos z - 1$

Calculation of the curl:

$$\boldsymbol{\nabla}\times\mathbf{F} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}}$$

a) $\boldsymbol{\nabla}\times\mathbf{F} = (0-0)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (y - -1)\hat{\mathbf{z}} = (y+1)\hat{\mathbf{z}}$

b) $\boldsymbol{\nabla}\times\mathbf{F} = (0-0)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (ye^{xy} - 1)\hat{\mathbf{z}} = (ye^{xy} - 1)\hat{\mathbf{z}}$

c) $\boldsymbol{\nabla}\times\mathbf{F} = (-1 - -xy^2\sin z)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (y^2\cos z - 2xe^x)\hat{\mathbf{z}}$

10) The gravitational field can be defined in terms of the gravitational potential as $\mathbf{g} = -\nabla\phi$. Obtain the gravitational field for a point mass whose gravitational potential is, in spherical coordinates:

$$\phi = -\frac{GM}{r}$$

Solution: We need to calculate the gradient of this potential. Remember the gradient is a vector.

$$\mathbf{g} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = -\left(\frac{\partial}{\partial x}\left(-\frac{GM}{r}\right), \frac{\partial}{\partial y}\left(-\frac{GM}{r}\right), \frac{\partial}{\partial z}\left(-\frac{GM}{r}\right)\right)$$
$$= GM\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \frac{\partial}{\partial y}\left(\frac{1}{r}\right), \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right)$$

In spherical coordinates, $r = \sqrt{x^2 + y^2 + z^2}$ so that $\frac{1}{r} = (x^2 + y^2 + z^2)^{-1/2}$

Hence, $\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{2}(2x)(x^2 + y^2 + z^2)^{-3/2}$, and similarly for the other components.

$$\mathbf{g} = -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z)$$

We can convert this to spherical coordinates and spherical basis. We know that $(x, y, z) = \mathbf{r} = r\hat{\mathbf{e}}_r$ and $x^2 + y^2 + z^2 = r^2$, therefore:

$$\mathbf{g} = -\frac{GM}{r^3} r\hat{\mathbf{e}}_r = -\frac{GM}{r^2}\hat{\mathbf{e}}_r$$

Which is the known gravitational field of a point mass.

11) A vector field is given by $\mathbf{F}(\mathbf{r}) = (y^2 - 2xyz^3, 3 + 2xy - x^2z^3, 6z^3 - 3x^2yz^2)$. Verify whether it derives from a potential and find the potential if it exists.

Solution: To verify whether it derives from a potential, we need to check that it is curl-free:

$$\mathrm{curl}(\mathbf{F}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}}$$
$$= (-3x^2z^2 + 3x^2z^2)\hat{\mathbf{x}} + (-6xyz^2 + 6xyz^2)\hat{\mathbf{y}} + (2y - 2xz^3 - 2y + 2xz^3)\hat{\mathbf{z}} = \mathbf{0}$$

Now we need to find the potential by performing a line integral of $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ along a path from $(0,0,0)$ to (x, y, z).

$$\int_0^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = \phi(\mathbf{0}) - \phi(\mathbf{r})$$

I have used $\mathbf{r}' = (x', y', z')$ for the variable being integrated, to distinguish it from the end-point of integration $\mathbf{r}$ which will be the coordinate of the potential. So, we can calculate the potential at any point (up to an arbitrary additive constant):

$$\phi(\mathbf{r}) = \underbrace{\phi(\mathbf{0})}_{\substack{\text{arbitrary} \\ \text{constant}}} - \int_0^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}'$$

Since the field is conservative, all paths of this line integral will give the same result. We can choose the most convenient path in which only one variable changes at a time: $(0,0,0) \to (x, 0,0) \to (x, y, 0) \to (x, y, z)$ so that:

$$\phi(x, y, z) = \phi_O - \int_{(0,0,0)}^{(x,0,0)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{(x,0,0)}^{(x,y,0)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{(x,y,0)}^{(x,y,z)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}'$$

Let's perform the integrals.

Long method: Remember the general method for calculation of line integrals after parametrizing the curve $\mathbf{r}(u)$. So let's parametrize each one:

$$\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_L \mathbf{F} \cdot \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\,\mathrm{d}u\right) = \int_a^b \big(\mathbf{F}(\mathbf{r}(u)) \cdot \boldsymbol{\tau}(u)\big)\,\mathrm{d}u$$

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \begin{pmatrix} \mathrm{d}x/\mathrm{d}u \\ \mathrm{d}y/\mathrm{d}u \\ \mathrm{d}z/\mathrm{d}u \end{pmatrix}$$

First integral: parametrize the curve as $\mathbf{r}' = (x', y', z') = (u, 0,0)$, so that $\mathrm{d}\mathbf{r}' = (\mathrm{d}u, 0,0)$

$$\int_{(0,0,0)}^{(x,0,0)} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = \int_0^x \mathbf{F}(u, 0,0) \cdot (\mathrm{d}u\,\hat{\mathbf{x}}) = \int_0^x F_x(u, 0,0)\,\mathrm{d}u = \int_0^x (y^2 - 2xyz^3)_{\substack{x=u \\ y=z=0}}\,\mathrm{d}u = 0$$

Second integral: parametrize the curve as $\mathbf{r}' = (x', y', z') = (x, u, 0)$, so that $\mathrm{d}\mathbf{r}' = (0, \mathrm{d}u, 0)$

$$\int_{(x,0,0)}^{(x,y,0)} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = \int_0^y \mathbf{F}(x, u, 0) \cdot (\mathrm{d}u\, \hat{\mathbf{y}}) = \int_0^y (3 + 2xy - x^2z^3)_{\substack{x=x\\y=u\\z=0}} \mathrm{d}u = \int_0^y (3 + 2xu)\, \mathrm{d}u$$
$$= 3y + xy^2$$

Third integral: parametrize the curve as $\mathbf{r}' = (x', y', z') = (x, y, u)$, so that $\mathrm{d}\mathbf{r}' = (0,0, \mathrm{d}u)$

$$\int_{(x,y,0)}^{(x,y,z)} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = \int_0^z \mathbf{F}(x, y, u) \cdot (\mathrm{d}u\, \hat{\mathbf{z}}) = \int_0^z (6z^3 - 3x^2yz^2)_{\substack{x=x\\y=y\\z=u}} \mathrm{d}u = \int_0^z (6u^3 - 3x^2yu^2)\, \mathrm{d}u$$
$$= \frac{3}{2}z^4 - x^2yz^3$$

Therefore:

$$\phi(x, y, z) = \phi_O - 0 - (3y + xy^2) - \left(\frac{3}{2}z^4 - x^2yz^3\right)$$

$$\phi(x, y, z) = \phi_O - 3y - xy^2 - \frac{3}{2}z^4 + x^2yz^3$$

You can now check that this potential gives rise to the field by calculating its gradient.

Note that I went down the robust but long path, individually parametrizing each curve, to avoid any confusion between the variable of integration and the variables being held constants on each path.

Once we have gained confidence doing line integrals, this whole problem could be done in three lines:

<u>Fast method</u>:

$$\phi(x, y, z) = \phi_O - \int_{(0,0,0)}^{(x,0,0)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{(x,0,0)}^{(x,y,0)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{(x,y,0)}^{(x,y,z)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}'$$
$$= \phi_O - \int_{(0,0,0)}^{(x,0,0)} F_x\, \mathrm{d}x - \int_{(x,0,0)}^{(x,y,0)} F_y\, \mathrm{d}y - \int_{(x,y,0)}^{(x,y,z)} F_z\, \mathrm{d}z$$
$$= \phi_O - \int_0^x \underbrace{(y^2 - 2xyz^3)}_{\substack{x=x\\y=0\\z=0}} \mathrm{d}x - \int_0^y \underbrace{(3 + 2xy - x^2z^3)}_{\substack{x=x\text{ (const)}\\y=y\\z=0}} \mathrm{d}y - \int_0^z \underbrace{(6z^3 - 3x^2yz^2)}_{\substack{x=x\text{ (const)}\\y=y\text{ (const)}\\z=z}} \mathrm{d}z$$
$$= \phi_O - \underbrace{\int_0^x 0\, \mathrm{d}x}_{0} - \underbrace{\int_0^y (3 + 2xy)\mathrm{d}y}_{3y+xy^2} - \underbrace{\int_0^z (6z^3 - 3x^2yz^2)\mathrm{d}z}_{\frac{3}{2}z^4 - x^2yz^3}$$

12) Use the previous result to find the line integral $\int_{(0,0,0)}^{(1,2,1)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ along any path from $(x, y, z) = (0,0,0)$ to $(1,2,1)$, where $\mathbf{F}$ is the vector field from the previous problem.

Since the field derives from a potential $\mathbf{F} = -\nabla\phi$, the line integral is given by:

$$\int_{(0,0,0)}^{(1,2,1)} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \phi(0,0,0) - \phi(1,2,1) = [\phi_0] - \left[\phi_O - 3(2) - (1)(2)^2 - \frac{3}{2}(1)^4 + (1)^2(2)(1)^3\right]$$
$$= 15/2$$

13) A vector field is given by $\mathbf{F}(\mathbf{r}) = (6xy + 2z^2, 3x^2 + 3z, 4xz + 3y)$. Verify whether it derives from a potential and find the potential if it exists.

Solution: To verify whether it derives from a potential, we need to check that it is curl-free:

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}}$$
$$= (3-3)\hat{\mathbf{x}} + (4z - 4z)\hat{\mathbf{y}} + (6x - 6x)\hat{\mathbf{z}} = \mathbf{0}$$

To obtain the potential, again we use the line integral $\int_0^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot \mathrm{d}\mathbf{r}' = \phi(\mathbf{0}) - \phi(\mathbf{r})$ to find:

(we use the fast method)

$$\begin{aligned}
\phi(x,y,z) &= \phi_O - \int_{(0,0,0)}^{(x,0,0)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{(x,0,0)}^{(x,y,0)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{(x,y,0)}^{(x,y,z)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\
&= \phi_O - \int_{(0,0,0)}^{(x,0,0)} F_x \,\mathrm{d}x - \int_{(x,0,0)}^{(x,y,0)} F_y \,\mathrm{d}y - \int_{(x,y,0)}^{(x,y,z)} F_z \,\mathrm{d}z \\
&= \phi_O - \int_0^x \underbrace{(6xy + 2z^2)}_{\substack{x=x\\y=0\\z=0}} \mathrm{d}x - \int_0^y \underbrace{(3x^2 + 3z)}_{\substack{x=x\text{ (const)}\\y=y\\z=0}} \mathrm{d}y - \int_0^z \underbrace{(4xz + 3y)}_{\substack{x=x\text{ (const)}\\y=y\text{ (const)}\\z=z}} \mathrm{d}z \\
&= \phi_O - \underbrace{\int_0^x 0\,\mathrm{d}x}_{0} - \underbrace{\int_0^y (3x^2)\mathrm{d}y}_{3x^2y} - \underbrace{\int_0^z (4xz + 3y)\mathrm{d}z}_{2xz^2+3yz} \\
&= \phi_O - 3x^2y - 2xz^2 - 3yz
\end{aligned}$$

14) Show that the magnetic field created by a z-directed current $\mathbf{I} = I_0\hat{\mathbf{z}}$ crossing through the origin, which is given below, is solenoidal:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I_1}{2\pi\rho}\hat{\mathbf{e}}_\phi$$

Solution: To prove that a field is solenoidal, we need to check that $\nabla \cdot \mathbf{B} = 0$ (i.e. it is divergence free)

Two options:

(1) **Calculate it in cartesian basis**.

We first need to convert $\mathbf{B}(\mathbf{r})$ into cartesian basis and coordinates. This is done via the change $\rho = \sqrt{x^2+y^2}$ and $\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}} = \frac{-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}$. Note that $\sin\phi$ and $\cos\phi$ can be easily found by drawing the right-angled triangle of sides x and y.

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I_1}{2\pi\sqrt{x^2+y^2}}\left(\frac{-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}\right) = \frac{\mu_0 I\ (-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}})}{2\pi(x^2+y^2)}$$

For which we can now calculate the divergence with our usual method:

$$\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial}{\partial x}\left(\frac{-y\mu_0 I}{2\pi(x^2+y^2)}\right) + \frac{\partial}{\partial y}\left(\frac{x\mu_0 I}{2\pi(x^2+y^2)}\right)$$

$$= (-2x)\frac{-y\mu_0 I}{2\pi(x^2+y^2)^2} + (-2y)\frac{x\mu_0 I}{2\pi(x^2+y^2)^2} = 0$$

(2) **Calculate it directly in cylindrical basis**. This is much easier but is unfortunately out of the scope of this module. I only show it here for completeness. Divergence in cylindrical coordinates can be shown to be:

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B}{\partial z}$$

(we usually look up this equation in a book, or online, for instance Wikipedia has a nice table for divergence and curl in cylindrical and spherical coordinates:

https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates)

Since this magnetic field only has B_ϕ component, the divergence is $\nabla \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} = \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{\mu_0 I_1}{2\pi\rho}\right) = 0$.

6. ORDINARY DIFFERENTIAL EQUATIONS

Nature seems to write its laws in the form of differential equations. Ordinary differential equations use ordinary derivatives (as opposed to partial derivatives) and therefore involve an unknown function $y(x)$. Let's start by reviewing what you learnt last semester: some concepts can be understood better thanks to our knowledge of linear algebra.

A. REVIEW OF ODE CLASSIFICATION

Order: The order of the ODE equals the order of the highest-order derivative appearing in it.

The general solution of an n-th order ODE contains n arbitrary parameters (constants). These must be determined by providing n externally imposed (boundary) conditions.

Degree: Highest power of the highest order term (after fractional powers are removed)

B. REVIEW OF SOME METHODS TO SOLVE ODEs OF ORDER 1

DIRECT INTEGRATION:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \rightarrow \mathrm{d}y = f(x)\mathrm{d}x \rightarrow \int \mathrm{d}y = \int f(x)\mathrm{d}x \rightarrow y = \int f(x)\mathrm{d}x$$

SEPARATION OF VARIABLES:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \rightarrow \frac{1}{g(y)}\mathrm{d}y = f(x)\mathrm{d}x \rightarrow \int \frac{1}{g(y)}\mathrm{d}y = \int f(x)\mathrm{d}x$$

C. REVIEW OF LINEAR ODEs

An ODE is **linear** if it can be written as: $\sum f_i(x)y^{(i)}(x) = r(x)$

where we used the notation $y^{(i)} = \frac{\mathrm{d}^i y}{\mathrm{d}x^i}$.

Linear ODEs fulfil that if y_1 and y_2 are solutions, then $\lambda y_1 + \mu y_2$ is also a solution.

Linear ODEs **general solution is** given by $y(x) = y_C(x) + y_P(x)$, where:

- $y_C(x)$ is the **complementary solution** (solution to the homogeneous system $\sum f_i(x)y^{(i)}(x) = 0$ which will be a linear combination of n terms each with arbitrary amplitude, corresponding to the free parameters or dimension of the null space).
- $y_P(x)$ is the **particular solution** (any solution to the whole system $\sum f_i(x)y^{(i)}(x) = r(x)$ as long as it is linearly independent to the $y_C(x)$)
- This is completely analogous to the solution of $\mathbf{Ay} = \mathbf{c}$ given by:
  $\mathbf{y} = \underbrace{\mathrm{span}\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N\}}_{\text{sol. to } \mathbf{Ay}=\mathbf{0}} + \mathbf{y}_p$

D. LINEAR ODEs WITH CONSTANT COEFFICIENTS

$$\sum a_i y^{(i)}(x) = r(x)$$

$$a_N \frac{\mathrm{d}^N y}{\mathrm{d}x^N} + \cdots + a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = r(x)$$

Easy procedure for solving this type of equation:

- Obtain $y_C(x)$ by setting $r(x) = 0$ (solution to the homogeneous equation) and using the ansatz $y(x) = Ae^{mx}$. This reduces the homogeneous differential equation into an algebraic equation (**characteristic polynomial** $p(m) = 0$) that can be solved for m. Each root of m provides an independent solution with an arbitrary scaling coefficient c_i.

$$y_C(x) = c_1 e^{m_1 x} + \cdots + c_N e^{m_N x}$$

  - If any of the roots is **repeated** k times, then use $Ae^{mx}, Bxe^{mx}, \ldots, Cx^{k-1}e^{mx}$ as k independent solutions (i.e. multiply by x as many times as needed to get k terms).

- Obtain $y_P(x)$ by looking at the form of $r(x)$ and using a similar form as ansatz for $y(x)$ (see table below - **method of undetermined coefficients**) with arbitrary coefficients to be determined by substitution on the ODE:

$r(x) = \text{Polynomial degree } n$	$\rightarrow$	$y_P(x) = \text{Polynomial degree } n \text{ with coefs } A_i$
$r(x) = \cos(x) \text{ or } \sin(x)$	$\rightarrow$	$y_P(x) = A\cos(x) + B\sin(x)$
$r(x) = Ce^{kx}$	$\rightarrow$	$y_P(x) = Ae^{kx}$
$r(x) = \text{Product of above}$	$\rightarrow$	$y_P(x) = \text{Product of above}$

  - The particular solution $y_P(x)$ **must be linearly independent** to the complementary solution $y_C(x)$. If the suggested particular solution has terms which already exist in the complementary solution, then multiply the entire particular solution by the **smallest integer power of x which ensures that none of the resulting terms appears in $y_C(x)$.**

1) Solve the ODE: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

Solution: It is a linear ODE with constant coefficients, so we need to find $y_C(x)$ and $y_P(x)$.

Solving the homogeneous equation to find the complementary solution:

$y_C(x)$ is the solution to the homogeneous equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$. Solved with the ansatz $y(x) = Ae^{mx}$. Substituting into the homogeneous equation we arrive at the characteristic polynomial $m^2 - 2m + 1 = 0$, which can be factorised as $(m-1)(m-1) = 0$. Therefore $m = 1$ is a repeated root! So, the complementary solution is: $y_C(x) = A_1e^x + A_2xe^x$.

Finding the particular solution:

$y_P(x)$ is the particular solution, found by looking at the form of $r(x) = e^x$, which seems to suggest us to try the ansatz $y(x) = Ae^x$.

However, this solution is already "taken" by the complementary solution! We have to try $y(x) = Ax^2e^{kx}$ because Ae^{kx} and Axe^{kx} are already "taken". (What do I mean by "taken"? Note that any combination of those two terms, $Ae^{kx} + Bxe^{kx}$ if substituted on the left-hand side of the differential equation, would give zero, as that is the definition for how we found them in the first place. So, they cannot be a particular solution).

Substituting $y(x) = Ax^2e^{kx}$ into the complete ODE (requires applying product rule several times) we find:

$$A\left(\frac{d}{dx}(x^2e^x + 2xe^x) - 2(x^2e^x + 2xe^x) + x^2e^x\right) = e^x$$

$$A\big((2xe^x + x^2e^x) + (2e^x + 2xe^x) - 2(x^2e^x + 2xe^x) + x^2e^x\big) = e^x$$

$$A(\cancel{2x} + \cancel{x^2} + 2 + \cancel{2x} - \cancel{2x^2} - \cancel{4x} + \cancel{x^2})e^x = e^x$$

$$2Ae^x = e^x \rightarrow A = (1/2)$$

General solution:

$$y(x) = y_C(x) + y_P(x)$$

$$y(x) = A_1e^x + A_2xe^x + \frac{1}{2}x^2e^x$$

E. SOME METHODS FOR SOLVING FIRST ORDER ODEs

First order first degree differential equations can be always re-written as:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = 0$$

I. EXACT DIFFERENTAL EQUATIONS:

In some cases, we might be lucky that the left-hand side of the ODE **happens to be the total differential of a function** $F(x,y)$ [remember the definition of $\mathrm{d}F$ from partial derivatives].

$$\underbrace{\underbrace{A(x,y)}_{\frac{\partial F}{\partial x}}\mathrm{d}x + \underbrace{B(x,y)}_{\frac{\partial F}{\partial y}}\mathrm{d}y}_{\mathrm{d}F} = 0$$

When this happens, the ODE is called an **exact differential equation**, and $A(x,y)\,\mathrm{d}x + B(x,y)\mathrm{d}y$ is called an **exact differential**. The ODE can be written as $\mathrm{d}F = 0$. The solution is trivial, by integrating both sides:

$$F(x,y) = \text{constant}$$

which can be considered as a solution to our equation (as it contains no derivatives) even though it is not written in the form $y = y(x)$.

To check if an ODE is exact, we need to verify whether a function $F(x,y)$ exists such that:

$$\begin{cases} A(x,y) = \partial F/\partial x \\ B(x,y) = \partial F/\partial y \end{cases} \qquad (1)$$

The quickest way to check this is to **test the equality of the cross partial derivatives** $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$, which in accordance to (1) implies the condition:

$$\boxed{\text{ODE is exact} \quad \Leftrightarrow \quad \frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}} \qquad (2)$$

If Eq. (2) is fulfilled, then we know that we can find a function $F(x,y)$ such that $\mathrm{d}F = A(x,y)\,\mathrm{d}x + B(x,y)\mathrm{d}y$. There are two methods to find it:

A) We can integrate either of the two equations in (1). For example, integrating the first one:

$$A(x,y) = \frac{\partial F}{\partial x} \quad \rightarrow \quad F(x,y) = \int A(x,y)\ \mathrm{d}x + c(y)$$

where **the "arbitrary constant" may still (and most probably will) be a function of the other variable** y. The unknown function $c(y)$ is obtained from the other condition in (1), that is, $B(x,y) = \partial F/\partial y$.

B) Use the technique we used to obtain the potential of a conservative field:

$$\int_{(0,0)}^{(x,y)} A(x,y)\,\mathrm{d}x + B(x,y)\mathrm{d}y = \int_{(0,0)}^{(x,y)} \mathrm{d}F = F(x,y) - F(0,0)$$

choosing a simple path of integration $(0,0) \rightarrow (x,0) \rightarrow (x,y)$.

2) Example: Solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy}{x^2+y^2}$$

Solution: This first order ODE is not easily separable. We can try to see if it is an exact differential ODE. We first rewrite it as:

$$\underbrace{2xy}_{A(x,y)}\ \mathrm{d}x + \underbrace{(x^2+y^2)}_{B(x,y)}\ \mathrm{d}y = 0$$

Now we check for the condition that this equation is exact:

$$\frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$$

$$2x = 2x$$

Indeed, it is fulfilled, so the ODE is exact! Excellent news; there exists a function $F(x,y)$ such that:

$$\begin{cases} A(x,y) = \partial F/\partial x \\ B(x,y) = \partial F/\partial y \end{cases} \Longrightarrow\ A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = \mathrm{d}F$$

Let's find it. **Method 1**: We may integrate $\int A\,\mathrm{d}x$ or $\int B\,\mathrm{d}y$. Let's try the first:

$$F(x,y) = \int A\,\mathrm{d}x = \int 2xy\,\mathrm{d}x = x^2y + c_1(y)$$

The "arbitrary constant" can be found by the second condition:

$$B(x,y) = \frac{\partial F}{\partial y}$$

$$x^2 + y^2 = x^2 + \frac{\partial c_1(y)}{\partial y}$$

From which we find $c_1(y) = \frac{1}{3}y^3 + c_2$. Therefore, $F(x,y) = x^2y + \frac{1}{3}y^3 + c_2$.

Method 2: We can think of a line integral between $(0,0)$ and (x,y):

$$\int_{(0,0)}^{(x,y)} A(x,y)\,\mathrm{d}x + B(x,y)\mathrm{d}y\ = \int_{(0,0)}^{(x,y)} \mathrm{d}F = F(x,y) - \underbrace{F(0,0)}_{c_2}$$

Hence: $F(x,y) = \left(\int_{(0,0)}^{(x,y)} A(x,y)\,\mathrm{d}x + B(x,y)\mathrm{d}y\right) + c_2$

So, we choose a simple path $(0,0) \to (x,0) \to (x,y)$.

In the first leg of the path, $\mathrm{d}y = 0$ and $y = 0$. In the second leg, $\mathrm{d}x = 0$ and x is a constant.

$$\int_{(0,0)}^{(x,y)} A(x,y)\,\mathrm{d}x + B(x,y)\mathrm{d}y = \int_0^x A(x,0)\mathrm{d}x + \int_0^y B(x,y)\mathrm{d}y = \int_0^x \underbrace{2xy}_{y=0}\ \mathrm{d}x + \int_0^y \underbrace{(x^2+y^2)}_{\substack{x=\text{const}\\ y=y}}\ \mathrm{d}y$$

$$= \left[x^2y + \frac{1}{3}y^3\right]_0^y = x^2y + \frac{1}{3}y^3.$$

So, the exact ODE can be written as

$$\underbrace{2xy}_{\partial F/\partial x}\ \mathrm{d}x + \underbrace{(x^2 + y^2)}_{\partial F/\partial y}\ \mathrm{d}y = 0$$

$$\mathrm{d}F(x, y) = 0$$

Integrating both sides:

$$F(x, y) = c_0$$

$$x^2 y + \frac{1}{3} y^3 = A$$

where $A = c_0 - c_2$ is an arbitrary constant determined by external (boundary) conditions.

As usual, remember that $y = y(x)$:

$$x^2 y(x) + \frac{1}{3} y(x)^3 = A$$

Notice that we obtained a valid condition for the function $y(x)$ but we **do not have an explicit expression for that function**. We can **check our answer** by taking the derivative with respect to x at both sides. Remember that $y = y(x)$ and so we must use the product rule and chain rule.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y(x) + \frac{1}{3} y(x)^3\right) = 0$$

$$\left(2xy + x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) + y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$2xy\,\mathrm{d}x + (x^2 + y^2)\mathrm{d}y = 0$$

Which gave us back the original ODE.

Alternative way of understanding exact ODEs:

When the left-hand-side is exact, divide the ODE by $\mathrm{d}x$ to put it in the form containing $\mathrm{d}y/\mathrm{d}x$:

$$\frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y = 0$$
$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

In that form, the left-hand side is the derivative with respect to x of some function $F(x,y)$ [just recognising the chain rule for the total derivative $\mathrm{d}F/\mathrm{d}x$, remembering that $y = y(x)$ and so we need to use the chain rule on y]. So we can write:

$$\frac{\mathrm{d}}{\mathrm{d}x}(F) = 0$$

$$\boxed{\text{ODE is exact if it can be written as } \frac{\mathrm{d}}{\mathrm{d}x}(F) = 0}$$

Sometimes this is easy to recognise by remembering common derivatives of functions $F(x,y)$. e.g. we know that $(x^2 + y^2)' = 2x + 2yy'$, therefore if we see $2x + 2yy'$ on the left-hand side of an ODE, we know it is exact and know how to solve it:

- Examples:

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = 0 \quad \rightarrow \quad x^2 + y^2 = C$$

$$y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(xy) = 0 \quad \rightarrow \quad y = C/x$$

$$\frac{1}{x^2}\left(y - x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = 0 \quad \rightarrow \quad y = Cx$$

$$e^{\frac{1}{2}x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + xe^{\frac{1}{2}x^2}y = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{1}{2}x^2}y\right) = 0 \quad \rightarrow \quad e^{\frac{1}{2}x^2}y = C$$

In all these cases $A + B\frac{dy}{dx}$, the previous test $\frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$ is still a valid check for the exactness of the ODE.

EXACT DIFFERENTIAL EQUALLING A FUNCTION OF x INSTEAD OF ZERO

An ODE can be written as an **exact differential equalling a function of x instead of zero**! In that case, we can still integrate both sides:

- Examples (same as above, but with non-zero right-hand side):

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = \sin x \quad \rightarrow \quad x^2 + y^2 = \underbrace{\int \sin x \ \mathrm{d}x}_{\cos x} + C$$

$$y + x\frac{\mathrm{d}y}{\mathrm{d}x} = e^{2x} \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(xy) = e^{2x} \quad \rightarrow \quad xy = \frac{e^{2x}}{2} + C \quad \rightarrow \quad y = \frac{e^{2x}}{2x} + \frac{C}{x}$$

II. INTEGRATING FACTORS

In some cases, an ODE is **not exact as it does not fulfil the condition:**

$$\text{ODE is not exact} \quad \Leftrightarrow \quad \frac{\partial B}{\partial x} \neq \frac{\partial A}{\partial y}$$

$$\underbrace{A(x,y)}_{\neq \frac{\partial F}{\partial x}} \mathrm{d}x + \underbrace{B(x,y)}_{\neq \frac{\partial F}{\partial y}} \mathrm{d}y = 0$$

However,

Sometimes we can multiply the ODE by a function $I(x,y)$ called an **integrating factor**, such that the resulting ODE **is** exact:

$$\underbrace{I(x,y)A(x,y)}_{=\frac{\partial F}{\partial x}} \mathrm{d}x + \underbrace{I(x,y)B(x,y)}_{=\frac{\partial F}{\partial y}} \mathrm{d}y = 0$$

$$\text{New ODE is exact} \quad \Leftrightarrow \quad \frac{\partial}{\partial x}(IB) = \frac{\partial}{\partial y}(IA)$$

And we can now solve this exact ODE.

Unfortunately, **finding the correct integrating factor requires in general inspiration or guesswork**. You can test broad cases of $I(x,y)$ using unknown parameters, and then use the condition $\frac{\partial}{\partial x}(IB) = \frac{\partial}{\partial y}(IA)$ to find which values of the parameters fulfil the condition.

3) Solve the equation

$$y - x\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Solution: Rewrite the equation as:

$$y\,\mathrm{d}x + (-x)\,\mathrm{d}y = 0$$

This is not exact as $\frac{\partial(y)}{\partial y} \neq \frac{\partial(-x)}{\partial x}$.

However, if we multiply times the integration factor $I(x,y) = \frac{1}{x^2}$ in both sides, then the equation becomes:

$$\frac{y}{x^2}\,\mathrm{d}x + \left(-\frac{1}{x}\right)\mathrm{d}y = 0$$

This equation is now exact: $\frac{\partial}{\partial y}\left(\frac{y}{x^2}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{x}\right)$

So, we can find a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = \frac{y}{x^2}$ and $\frac{\partial F}{\partial y} = -\frac{1}{x}$.

To find $F(x,y)$ we can follow one of the two methods discussed earlier, or guess it:

$$F(x,y) = -\frac{y}{x}$$

Therefore, the solution to the ODE is:

$$F(x,y) = -\frac{y}{x} = A$$

$$y = Cx$$

But how did we know to use the factor $I(x,y) = \frac{1}{x^2}$ in the first place?

One could argue that, seeing $y - xy' = 0$, you identify something that reminds you to the quotient rule $\left(\frac{y}{x}\right)' = \frac{y-xy'}{x^2}$ but it is missing the $\frac{1}{x^2}$, so you add it as an integrating factor! which automatically turns the ODE into $\left(\frac{y}{x}\right)' = 0$, an exact equation, whose solution is $\frac{y}{x} = C$.

Alternatively, we could have guessed a much more general integrating factor $I(x,y) = x^\alpha y^\beta$ and then looked for values of α and β which fulfil the exact ODE condition:

$$\frac{\partial}{\partial y}(IA) = \frac{\partial}{\partial x}(IB)$$

$$\frac{\partial}{\partial y}\left(x^\alpha y^\beta y\right) = \frac{\partial}{\partial x}\left(x^\alpha y^\beta(-x)\right)$$

$$(\beta+1)\left(x^\alpha y^\beta\right) = -(\alpha+1)\left(x^\alpha y^\beta\right)$$

A possible solution is $\beta = 0$ and $\alpha = -2$, which means $I(x,y) = \frac{1}{x^2}$, precisely the one we used.

III. FIRST ORDER LINEAR ODEs – INTEGRATING FACTOR CAN BE KNOWN FOLLOWING A RECIPE

When a first order ODE is linear it can be written as:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

Then, it can be shown that **the integrating factor** $I(x) = \exp\{\int P(x)\mathrm{d}x\}$ **always** converts the left-hand side of the equation into **an exact differential**!

Proof: Multiplying the ODE by the integrating factor:

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)P(x)y = I(x)Q(x) \quad (1)$$

If the integrating factor makes the equation exact, then we know it can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}x}[F(x)] = I(x)Q(x)$$

But, looking at (1), the first term on the left-hand side seems to be the first term in the product rule of $\frac{\mathrm{d}}{\mathrm{d}x}(I(x)y(x))$. Indeed, if we assume that $F(x) = I(x)y(x)$ then we get:

$$I\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}I}{\mathrm{d}x}y = I(x)Q(x)$$

Comparing this with Eq. (1) we see that we can have exactly this form if $\frac{\mathrm{d}I(x)}{\mathrm{d}x} = P(x)I(x)$ which is fulfilled by an exponential integrating factor $I(x) = \exp\{\int P(x)\mathrm{d}x\}$

$$\underset{\text{1st order linear ODE}}{\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)} \Leftrightarrow \begin{array}{c}\text{ODE can be made exact with}\\ \text{a known integrating factor}\end{array} \quad I(x) = \exp\left\{\int P(x)\mathrm{d}x\right\}$$

4) Solve the linear ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 4x$$

Solution: Since it is linear first order ODE, $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$, we know that the integrating factor $I(x) = \exp\{\int P(x)\mathrm{d}x\}$ will always make the left-hand side exact.

$$I(x) = \exp\left\{\int 2x \,\mathrm{d}x\right\} = e^{x^2}$$

Indeed, multiplying both sides by the integrating factor:

$$e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xe^{x^2}y = 4xe^{x^2}$$

We know that this equation must now be exact, so the left-hand side must be the x derivative of a function $F(x,y)$ that we need to find. We can see that the left-hand side is nothing else than the product rule, so we can write:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{x^2}\right) = 4xe^{x^2}$$

Which can now be solved by integration:

$$ye^{x^2} = \int 4xe^{x^2}\,\mathrm{d}x$$

$$ye^{x^2} = 2e^{x^2} + C$$

$$y = 2 + Ce^{-x^2}$$

Notice that, since the ODE was a linear equation, the solution is of the expected form. The term with arbitrary scaling $y_H(x) = Ce^{-x^2}$ is the solution to the homogeneous equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 0$, while the term $y_P(x) = 2$ is a particular solution.

IV. HOMOGENEOUS DIFFERENTIAL EQUATIONS

As we know, some first order ODEs are separable, so all that depends on x can be put on one side, and all that depends on y on the other. They can be solved by integrating both sides.

$$a(x)\mathrm{d}x = b(y)\mathrm{d}y$$

Some ODEs are not separable initially but can be transformed into being separable by a change of variables. One example of this is that of "**homogeneous differential equations**". These are ODEs that can be written as $\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right)$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right)$ Homogeneous ODE	$\Leftrightarrow$	ODE can be solved with a change of variables	$z(x) = y/x$ $y = zx$

Proof: Doing this change of variables, you can see that: $y = z\,x \rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = z + x\frac{\mathrm{d}z}{\mathrm{d}x}$. Therefore:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(zx) = F(z)$$

$$z + x\frac{\mathrm{d}z}{\mathrm{d}x} = F(z)$$

which is separable:

$$\frac{\mathrm{d}z}{F(z) - z} = \frac{\mathrm{d}x}{x}$$

and can be solved by direct integration (in principle, if the integral can be done):

$$\int \frac{\mathrm{d}z}{F(z) - z} = \int \frac{\mathrm{d}x}{x}$$

5) Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

Solution: This is a homogeneous equation. We can solve it by the change of variables$y = z(x)\,x$.

$$y = zx \;\rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}x} = z + x\frac{\mathrm{d}z}{\mathrm{d}x}$$

Therefore:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \;\rightarrow\; z + x\frac{\mathrm{d}z}{\mathrm{d}x} = z + \tan z$$

Which is now a separable ODE:

$$\frac{\mathrm{d}z}{\tan z} = \frac{\mathrm{d}x}{x}$$

And can be solved by integration:

$$\int \frac{\cos z}{\sin z}\mathrm{d}z = \int \frac{\mathrm{d}x}{x}$$

$$\ln(\sin z) = \ln(x) + c_1$$

$$\sin z = Ax$$

$$\sin\left(\frac{y}{x}\right) = Ax$$

$$y = x\sin^{-1}(Ax)$$

Identifying homogeneous ODEs:

How can we know if an ODE is homogeneous $\frac{\mathrm{d}y}{\mathrm{d}x} = F(y/x)$ when it is written in the standard way?

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{A(x,y)}{B(x,y)}$$

The answer is that we can check if A and B are "*homogeneous functions of the same order*". A function $f(x,y)$ is said to be **homogeneous of order** n if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$.

$$\begin{array}{c}\begin{cases} A(\lambda x,\lambda y) = \lambda^n A(x,y) \\ B(\lambda x,\lambda y) = \lambda^n B(x,y)\end{cases} \\ \text{A and B are homogeneous order n}\end{array} \implies \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

Proof: Make the change of variables $y = zx$ in A and B, and use the fact that they are homogeneous:

$$\frac{A(x,y)}{B(x,y)} = \frac{A(x,xz)}{B(x,xz)} = \frac{x^n A(1,z)}{x^n B(1,z)} = \frac{A(1,z)}{B(1,z)} = F(z) = F\left(\frac{y}{x}\right)$$

In practice, we see that for both A and B to be homogeneous, and of the same degree, **we require the sum of the powers in x and y in each term of A and B to be the same**. Therefore, the homogeneity of the ODE can, in practice, be evaluated by simple inspection:

Order 1:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + 3y}{y}$$

Order 2:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 3yx}{y^2 + yx}$$

Order 3:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 + y^2x}{x^2y + y^3 + x^3}$$

When an homogeneous ODE is identified, we can apply the change of variables "blindly" without having to find the function $F(y/x)$, and the ODE will always be reduced to a separable one.

Proof:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = 0$$

Now we apply the change of variables $y = zx \quad \rightarrow \quad \mathrm{d}y = z\,\mathrm{d}x + x\,\mathrm{d}z$ and we get:

$$A(x,xz)\mathrm{d}x + B(x,xz)(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0$$

Using the fact that A and B are homogeneous:

$$x^n A(1,z)\mathrm{d}x + x^n B(1,z)(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0$$

$$A(1,z)\mathrm{d}x + B(1,z)(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0$$

which is separable:

$$\frac{\mathrm{d}x}{x} = -\frac{B(1,z)}{A(1,z) + z\,B(1,z)}\mathrm{d}z$$

And can be solved by integration.

6) Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 y}{x^3 + y^3}$$

Solution: In the unfolded form, it becomes:

$$\underbrace{(-x^2 y)}_{A(x,y)}\,\mathrm{d}x + \underbrace{(x^3 + y^3)}_{B(x,y)}\,\mathrm{d}y = 0$$

This turns out to be a homogeneous equation, because A and B are homogeneous of the same order.

$$A(\lambda x, \lambda y) = (\lambda x)^2(\lambda y) = \lambda^3(x^2 y) = \lambda^3 A(x, y)$$
$$B(\lambda x, \lambda y) = (\lambda x)^3 + (\lambda y)^3 = \lambda^3(x^3 + y^3) = \lambda^3 B(x, y)$$

Indeed, we can see that all terms in both A and B have a sum of powers of x and y equal to 3.

Therefore, the equation is homogeneous.

We have two possible paths now:

Path 1: Write the equation in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right)$, which we now know is possible when A and B are homogeneous, and then do the change of variables $z = y/x$ and solve as above.

Path 2: Directly do the change of variables $z = y/x$ in the unfolded form and solve.

Both paths give the same final integrals to do. Let's follow Path 2.

We do the change of variables $z = y/x \to y = zx \ \to \ \mathrm{d}y = z\,\mathrm{d}x + x\,\mathrm{d}z$

$$(-x^2 y)\mathrm{d}x + (x^3 + y^3)\mathrm{d}y = 0$$
$$(-x^3 z)\,\mathrm{d}x + (x^3 + x^3 z^3)(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0$$
$$(-z)\mathrm{d}x + (1 + z^3)(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0$$
$$\mathrm{d}x(z^4) + \mathrm{d}z\,x(1 + z^3) = 0$$
$$\frac{\mathrm{d}x}{x} = -\frac{(1 + z^3)}{z^4}\mathrm{d}z$$

Integrating both sides:

$$\int \frac{\mathrm{d}x}{x} = \int \left(-\frac{1}{z^4} - \frac{1}{z}\right)\mathrm{d}z$$

$$\ln(x) = \frac{1}{3z^3} - \ln(z) + c_1$$

$$\ln(x) + \ln(z) = \frac{1}{3z^3} + c_1$$

$$\ln(xz) = \frac{1}{3z^3} + c_1$$

And, substituting $z = y/x$ we get:

$$\ln(y) = \frac{x^3}{3y^3} + c$$

This is a transcendental equation in y (i.e. we cannot solve it as $y = y(x)$), and it is the general solution of the problem.

F. OTHER METHODS:

These were just an arbitrary selection of some methods for solving first order ODEs. Mathematics textbooks contain many other methods, e.g. [Riley, Hobson, Bence: Chapters 14 and 15], [K. A. Stroud Programmes 24 and 25]. I didn't mention methods using series solutions, e.g. [Riley, Hobson, Bence: Chapter 16].

As you can see different ODEs can be solved in different ways. Unfortunately, **there is no general method for solving all ODEs**. In fact, in general, **ODEs do not necessarily have a solution**.

Differential equations is a huge topic that would require entire modules to study. During your professional practice, when you are faced with a differential equation, this is my advice:

1) Check if it is a known differential equation:
   This happens very often if the ODE "looks" simple. It means that someone else solved it already! So, you can consult a book.

2) Try to solve it analytically:
   a. Use a symbolic mathematics software to try to find an analytic solution (e.g. Mathematica's DSolve function). Warning: this might fail or give you the answer in an unfamiliar notation.

```
DSolve[2 x + 2 y[x] y'[x] == Sin[x], y[x], x]
```

$$\left\{\left\{y[x] \to -\sqrt{-x^2 + 2\,C[1] - \mathrm{Cos}[x]}\right\}, \left\{y[x] \to \sqrt{-x^2 + 2\,C[1] - \mathrm{Cos}[x]}\right\}\right\}$$

```
DSolve[y'[x] == (y[x] / x) + Tan[y[x] / x], y[x], x]
```

$$\left\{\left\{y[x] \to x\,\mathrm{ArcSin}\left[e^{C[1]}\,x\right]\right\}\right\}$$

   a. Consult a book for known methods of solving ODEs to see if some method works

3) The differential equation might not have an analytic solution.
   Solve it computationally: We could have several modules' worth to say about this!

PROBLEMS

LINEAR ODEs WITH CONSTANT COEFFICIENTS

7) Solve

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4y=\sin 2x$$

Solution: It is a linear ODE with constant coefficients, so we need to find $y_C(x)$ and $y_P(x)$.

Solving the homogeneous equation:

$y_C(x)$ is the solution to the homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4y=0$. Solved with the ansatz $y(x)=Ae^{mx}$. Substituting into the homogeneous equation we arrive at the characteristic polynomial:

$m^2+4=0$, which can be factorised as $(m-2i)(m+2i)=0$. Therefore, the complementary solution is:

$$y_C(x)=A_1e^{2ix}+A_2e^{-2ix}=B_1\cos 2x+B_2\sin 2x$$

Solving the particular solution:

$y_P(x)$ is the particular solution, found by looking at the form of $r(x)=\sin 2x$, which seems to suggest us to try the ansatz:

$y_P(x)=\sin 2x+\cos 2x.$

However, the terms $\sin 2x$ and $\cos 2x$ are already "taken" by the complementary solution! Therefore, we must multiply by x as many times as needed to avoid the "taken" terms. In this case multiplying by x once will suffice:

$$y_P(x)=ax\sin 2x+bx\cos 2x$$

Now we can substitute this into the ODE in order to find the coefficients (a,b) [remember to use the product rule]:

$$\begin{aligned}y_P'(x)&=2ax\cos 2x+a\sin 2x-2bx\sin 2x+b\cos 2x\\&=(2ax+b)\cos 2x+(a-2bx)\sin 2x\end{aligned}$$

$$\begin{aligned}y_P''(x)&=-2(2ax+b)\sin 2x+2a\cos 2x+2(a-2bx)\cos 2x-2b\sin 2x\\&=(-4ax-4b)\sin 2x+(-4bx+4a)\cos 2x\end{aligned}$$

We can substitute $y_P(x)$ into the ODE: $y_P''(x)+4y_P(x)=\sin 2x$ and equate coefficients of $\sin 2x$ and $\cos 2x$ at both sides:

$$\begin{aligned}&(-4ax-4b)\sin 2x+(-4bx+4a)\cos 2x+4ax\sin 2x+4bx\cos 2x=\sin 2x\\&\to\ -4b\sin 2x+4a\cos 2x=\sin 2x\ \to\ \begin{cases}-4b=1\\4a=0\end{cases}\to\begin{cases}b=-1/4\\a=0\end{cases}\end{aligned}$$

$$y_P(x)=-\frac{1}{4}x\cos 2x$$

And so, the general solution reads: $y(x)=y_C(x)+y_P(x)$

$$y(x)=B_1\cos 2x+B_2\sin 2x-\frac{1}{4}x\cos 2x$$

8) Solve

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = e^x$$

Solution: It is a linear ODE with constant coefficients, so we need to find $y_C(x)$ and $y_P(x)$.

Solving the homogeneous equation:

$y_C(x)$ is the solution to the homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0$. Solved with the ansatz $y(x) = Ae^{mx}$. Substituting into the homogeneous equation we arrive at the characteristic polynomial:

$m^2 + m - 2 = 0$, which can be factorised as $(m+2)(m-1) = 0$. Therefore, the complementary solution is:

$$y_C(x) = A_1e^{-2x} + A_2e^x$$

Solving the particular solution:

$y_P(x)$ is the particular solution, found by looking at the form of $r(x) = e^x$, which seems to suggest us to try the ansatz $y_P(x) = ae^x$.

However, the term e^x is already "taken" by the complementary solution! Therefore, we have to multiply by x as many times as needed to avoid the taken terms. In this case multiplying by x once will suffice.

$$y_P(x) = axe^x$$

Now we can substitute this into the ODE in order to find the coefficient a.

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(axe^x) + \frac{\mathrm{d}}{\mathrm{d}x}(axe^x) - 2(axe^x) = e^x$$

$$a\,\frac{\mathrm{d}}{\mathrm{d}x}(xe^x + e^x) + a(xe^x + e^x) - 2axe^x = e^x$$

$$a\,(xe^x + e^x + e^x) + a(xe^x + e^x) - 2axe^x = e^x$$

$$3ae^x = e^x$$

$$a = \frac{1}{3}$$

So we find the particular solution to be:

$$y_P(x) = \frac{1}{3}xe^x$$

And so, the general solution reads:

$$y(x) = y_C(x) + y_P(x)$$

$$y(x) = A_1e^{-2x} + A_2e^x + \frac{1}{3}xe^x$$

EXACT DIFFERENTIAL EQUATIONS

9) Solve

$$xe^y \mathrm{d}y + e^y \mathrm{d}x = x^3 \, \mathrm{d}x$$

Solution: This is an equation in the form:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = f(x)$$

We can see that:

$$\frac{\partial B}{\partial x} = e^y = \frac{\partial A}{\partial y} = e^y$$

Therefore, the equation is exact, which means that $e^y \mathrm{d}x + xe^y \mathrm{d}y = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y = \mathrm{d}F$.

To find $\mathrm{d}F$, we can either think (it is not too difficult to realise that $F(x,y) = xe^y$) or apply one of two methods:

Method 1: Solve the simultaneous equations $A(x,y) = e^y = \frac{\partial F}{\partial x}$ and $B(x,y) = xe^y = \frac{\partial F}{\partial y}$ by integrating one and substituting the unknown coefficient on the other.

Method 2: Do a line integral of $e^y \mathrm{d}x + xe^y \mathrm{d}y = \mathrm{d}F$ along the path $(0,0) \to (x,0) \to (x,y)$

$$\int_{(0,0)}^{(x,y)} e^y \mathrm{d}x + xe^y \mathrm{d}y = \int_{(0,0)}^{(x,y)} \mathrm{d}F = F(x,y) - F_0$$

$$F(x,y) = F_0 + \int_{(0,0)}^{(x,y)} e^y \mathrm{d}x + xe^y \mathrm{d}y = F_0 + \int_0^x \underbrace{e^y}_{y=0} \mathrm{d}x + \int_0^y \underbrace{xe^y}_{x=const} \mathrm{d}y = F_0 + x + xe^y - x$$
$$= F_0 + xe^y$$

Therefore, the function $F(x,y) = F_0 + xe^y$ leads to the total differential on the LHS:

$$xe^y \mathrm{d}y + e^y \mathrm{d}x = x^3 \, \mathrm{d}x$$

$$\mathrm{d}F = x^3 \, \mathrm{d}x$$

Integrating both sides:

$$F = \frac{1}{4}x^4 + c$$

And substituting $F(x,y) = F_0 + xe^y$ (we may group the constants F_0 and c as one single $A = c - F_0$) we arrive at:

$$xe^y = \frac{1}{4}x^4 + A$$

$$e^y = \frac{1}{4}x^3 + \frac{A}{x}$$

$$y = \ln\left(\frac{1}{4}x^3 + \frac{A}{x}\right)$$

10) Solve

$$y\cos(xy)\,\mathrm{d}x + x\cos(xy)\,\mathrm{d}y = \mathrm{d}x$$

Solution: This is an equation in the form:

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = f(x)$$

We can see that:

$$\frac{\partial B}{\partial x} = \cos(xy) - xy\sin(xy) = \frac{\partial A}{\partial y} = \cos(xy) - xy\sin(xy)$$

Therefore, the equation is exact, which means that $y\cos(xy)\,\mathrm{d}x + x\cos(xy)\,\mathrm{d}y = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y = \mathrm{d}F$. To find $F(x,y)$, we can either think (it is not too difficult to realise that $F(x,y) = \sin(xy)$) or apply one of two methods:

Method 1: Solve the simultaneous equations $A(x,y) = y\cos(xy) = \frac{\partial F}{\partial x}$ and $B(x,y) = x\cos(xy) = \frac{\partial F}{\partial y}$ by integrating one and substituting the unknown coefficient on the other.

Method 2: Do a line integral of $e^y\mathrm{d}x + xe^y\mathrm{d}y = \mathrm{d}F$ along the path $(0,0) \to (x,0) \to (x,y)$

$$\int_{(0,0)}^{(x,y)} y\cos(xy)\,\mathrm{d}x + x\cos(xy)\,\mathrm{d}y = \int_{(0,0)}^{(x,y)} \mathrm{d}F = F(x,y) - F_0$$

$$\begin{aligned} F(x,y) &= F_0 + \int_{(0,0)}^{(x,y)} y\cos(xy)\,\mathrm{d}x + x\cos(xy)\,\mathrm{d}y = F_0 + \int_0^x \underbrace{y\cos(xy)}_{y=0}\,\mathrm{d}x + \int_0^y \underbrace{x\cos(xy)}_{x=const}\,\mathrm{d}y \\ &= F_0 + 0 + \sin(xy) \end{aligned}$$

Therefore, the function $F(x,y) = F_0 + \sin(xy)$ leads to the total differential on the LHS:

$$y\cos(xy)\,\mathrm{d}x + x\cos(xy)\,\mathrm{d}y = \mathrm{d}x$$

$$\mathrm{d}F = \mathrm{d}x$$

Integrating both sides:

$$F = x + c$$

And substituting $F(x,y) = F_0 + \sin(xy)$ (we may group the constants F_0 and c as one single $A = c - F_0$) we arrive at:

$$\sin(xy) = x + A$$

$$y = \frac{\sin^{-1}(x + A)}{x}$$

FIRST ORDER LINEAR ODE (NO CONSTANT COEFFICIENTS) OF THE FORM $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$.

11) Solve the first order linear ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2$$

This function **is first order linear, but not with constant coefficients**. It can be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

When an ODE can be written in this form, we know that an integrating factor $I(x) = \exp\{\int P(x)\mathrm{d}x\}$ **always** converts the left-hand side of the equation into an exact differential!

In this case, $P(x) = x^{-1}$ and $Q(x) = x^2$, hence the integrating factor must be:

$$I(x) = \exp\left\{\int P(x)\mathrm{d}x\right\} = \exp\left\{\int x^{-1}\mathrm{d}x\right\} = \exp\{\ln x\} = x$$

Multiplying times the integrating factor on both sides:

$$\frac{\mathrm{d}y}{\mathrm{d}x}x + y = x^3$$

We know that this must now be an exact differential equation, i.e.:

$$x\,\mathrm{d}y + \mathrm{d}x\,y = x^3\,\mathrm{d}x$$

Indeed, now the left-hand side is the exact differential of $F(x,y) = xy$. (If you could not find this easily, you can follow one of the two known methods discussed in the "exact differential equations" section, which involved integration). Therefore:

$$\mathrm{d}F = x^3\,\mathrm{d}x$$

So, we can integrate both sides:

$$F = \frac{1}{4}x^4 + c_1$$

And substituting $F = xy$, we get:

$$xy = \frac{1}{4}x^4 + c_1$$

$$y = \frac{1}{4}x^3 + \frac{c_1}{x}$$

Notice that, since the ODE was a linear equation, the solution is of the expected form. The term with arbitrary scaling $y_H(x) = \frac{c_1}{x}$ is the solution to the homogeneous equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = 0$, while the term $y_P(x) = \frac{1}{4}x^3$ is a particular solution.

12) Solve the first order linear ODE (not constant coefficients):

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{\tan x} y = \cos x$$

This function **is first order linear, but not with constant coefficients**. It can be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

When an ODE can be written in this form, we know that an integrating factor $I(x) = \exp\{\int P(x)\mathrm{d}x\}$ **always** converts the left-hand side of the equation into an exact differential!

In this case, $P(x) = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ and $Q(x) = \cos x$, hence the integrating factor must be:

$$I(x) = \exp\left\{\int P(x)\, \mathrm{d}x\right\} = \exp\left\{\int \frac{\cos x}{\sin x} \mathrm{d}x\right\} = \exp\{\ln(\sin x)\} = \sin x$$

Multiplying times the integrating factor on both sides:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\sin x + \frac{\sin x}{\tan x} y = \sin x \cos x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\sin x + y\cos x = \sin x \cos x$$

We know that this must now be an exact differential equation, i.e.:

$$\sin x\, \mathrm{d}y + y \cos x\, \mathrm{d}x = \sin x \cos x\, \mathrm{d}x$$

Indeed, the left-hand side is the exact differential of $F(x, y) = y \sin x$. (If you could not find this easily, you can follow one of the two known methods discussed in the “exact differential equations” section, which involved integration).

$$\mathrm{d}F = \sin x \cos x\, \mathrm{d}x$$

So we can integrate both sides:

$$F = \frac{1}{2}\sin^2 x + c_1$$

And substituting $F = y \sin x$, we get:

$$y \sin x = \frac{1}{2}\sin^2 x + c_1$$

$$y = \frac{1}{2}\sin x + \frac{c_1}{\sin x}$$

Notice that, since the ODE was a linear equation, the solution is of the expected form. The term with arbitrary scaling $y_H(x) = \frac{c_1}{\sin x}$ is the solution to the homogeneous equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{\tan x} y = 0$, while the term $y_P(x) = \frac{1}{2}\sin x$ is a particular solution.

HOMOGENEOUS DIFFERENTIAL EQUATIONS

13) Find the general solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+3y}{2x}$$

This ODE is not separable. Also, $(x+3y)\mathrm{d}x + 2x\,\mathrm{d}y$ is not an exact differential.

However, it is a **homogeneous equation**.

Both $A(x,y) = (x+3y)$ and $B(x,y) = 2x$ are homogeneous of order 1, because they fulfil:

$$\begin{cases} A(\lambda x, \lambda y) = \lambda^n A(x,y) \\ B(\lambda x, \lambda y) = \lambda^n B(x,y) \end{cases}$$

Therefore, the equation can be made separable by using the change of variables $y = zx$.

$$\mathrm{LHS} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(xz) = x\frac{\mathrm{d}z}{\mathrm{d}x} + z$$

$$\mathrm{RHS} = \frac{x+3y}{2x} = \frac{x+3xz}{2x} = \frac{1+3z}{2}$$

The resulting ODE must be separable:

$$x\frac{\mathrm{d}z}{\mathrm{d}x} + z = \frac{1+3z}{2}$$
$$x\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1+3z}{2} - z = \left(\frac{1}{2}\right)(1+z)$$

$$\frac{2}{1+z}\mathrm{d}z = \frac{1}{x}\mathrm{d}x$$

Integrating both sides:

$$2\ln(1+z) = \ln(x) + c_1$$

Taking the exponential on both sides ($A = e^{c_1}$):

$$(1+z)^2 = Ax$$

$$\left(1+\frac{y}{x}\right)^2 = Ax$$

$$(x+y)^2 = Ax^3$$

14) Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{xy}$$

This ODE is not separable. Also, $(x^2 + y^2)\mathrm{d}x + xy\,\mathrm{d}y$ is not an exact differential.

However, it is a **homogeneous equation**.

Both $A(x, y) = (x^2 + y^2)$ and $B(x, y) = xy$ are homogeneous of order 2, because they fulfil:

$$\begin{cases} A(\lambda x, \lambda y) = \lambda^n A(x, y) \\ B(\lambda x, \lambda y) = \lambda^n B(x, y) \end{cases}$$

Therefore, the equation can be made separable by using the change of variables $y = zx$.

$$\mathrm{LHS} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(xz) = x\frac{\mathrm{d}z}{\mathrm{d}x} + z$$

$$\mathrm{RHS} = \frac{x^2 + y^2}{xy} = \frac{x^2(1 + z^2)}{x^2 z} = \frac{1 + z^2}{z}$$

The resulting ODE must always be separable (we proved it for a general homogeneous ODE):

$$x\frac{\mathrm{d}z}{\mathrm{d}x} + z = \frac{1 + z^2}{z}$$

$$x\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1 + z^2}{z} - z = \frac{1}{z}$$

$$z\,\mathrm{d}z = \frac{1}{x}\mathrm{d}x$$

Integrating both sides:

$$\frac{1}{2}z^2 = \ln(x) + c_1$$

Substituting y back:

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln(x) + c_1$$

$$y^2 = 2x^2(\ln x + c_1)$$

MORE EXAMPLES (Linear Equations)

15) Find the form of the particular solution to this differential equation (make sure the particular solution is linearly independent to the complementary solution). You don't need to determine the coefficients of the particular solution because it is lots of algebra, just indicate the steps.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = x^2 \sin 2x$$

Solution: It is a linear ODE with constant coefficients, so we need to find $y_C(x)$ and $y_P(x)$.

Solving the homogeneous equation:

$y_C(x)$ is the solution to the homogeneous equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = 0$. Solved with the ansatz $y(x) = Ae^{mx}$. Substituting into the homogeneous equation we arrive at the characteristic polynomial:

$m^2 + 4 = 0$, which can be factorised as $(m - 2i)(m + 2i) = 0$. Therefore, the complementary solution is:

$$y_C(x) = A_1 e^{2ix} + A_2 e^{-2ix} = B_1 \cos 2x + B_2 \sin 2x$$

Solving the particular solution:

$y_P(x)$ is the particular solution, found by looking at the form of $r(x) = x^2 \sin 2x$, which seems to suggest us to try the ansatz:

$y_P(x) = (ax^2 + bx + c) \sin 2x + (dx^2 + ex + f) \cos 2x$.

However, the terms $\sin 2x$ and $\cos 2x$ are already "taken" by the complementary solution! Therefore, we have to multiply by x as many times as needed to avoid the taken terms. In this case multiplying by x once will suffice:

$$y_P(x) = (ax^3 + bx^2 + cx) \sin 2x + (dx^3 + ex^2 + fx) \cos 2x$$

Now we can substitute this into the ODE in order to find the coefficients (a, b, c, d, e, f). Lots of algebra! Double derivative requires doing the product rule and large equating coefficients problem. Anyway, after substitution of $y_P(x)$ into the ODE and equating coefficients, one can find the values of (a, b, c, d, e, f) and build the general solution as $y(x) = y_C(x) + y_P(x)$.

A computer can easily perform the algebra, so we find:

$$y_P(x) = -\frac{x^3}{12} \cos 2x + \frac{x^2}{16} \sin 2x + \frac{x}{32} \cos 2x$$

And so, the general solution reads: $y(x) = y_C(x) + y_P(x)$

$$y(x) = B_1 \cos 2x + B_2 \sin 2x - \frac{x^3}{12} \cos 2x + \frac{x^2}{16} \sin 2x + \frac{x}{32} \cos 2x$$